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A HISTORY OF MATHEMATICS

BOOKS BY FLORIAN CAJORI

HISTORY OF MATHEMATICS
Revised and Enlarged Edition

HISTORY OF ELEMENTARY MATHEMATICS
Revised and Enlarged Edition

HISTORY OF PHYSICS

INTRODUCTION TO THE MODERN
THEORY OF EQUATIONS

A HISTORY OF

MATHEMATICS

BY

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"I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history."—J. W. L. GLAISHER

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PREFACE TO THE SECOND EDITION

In preparing this second edition the earlier portions of the book have been partly re-written, while the chapters on recent mathematics are greatly enlarged and almost wholly new. The desirability of having a reliable one-volume history for the use of readers who cannot devote themselves to an intensive study of the history of mathematics is generally recognized. On the other hand, it is a difficult task to give an adequate bird's-eye-view of the development of mathematics from its earliest beginnings to the present time. In compiling this history the endeavor has been to use only the most reliable sources. Nevertheless, in covering such a wide territory, mistakes are sure to have crept in. References to the sources used in the revision are given as fully as the limitations of space would permit. These references will assist the reader in following into greater detail the history of any special subject. Frequent use without acknowledgment has been made of the following publications: *Annuario Biografico del Circolo Matematico di Palermo*, 1914; *Jahrbuch über die Fortschritte der Mathematik*, Berlin; *J. C. Poggendorff's Biographisch-Literarisches Handwörterbuch*, Leipzig; *Gedenktagebuch für Mathematiker*, von Felix Müller, 3. Aufl., Leipzig und Berlin, 1912; *Revue Semestrielle des Publications Mathématiques*, Amsterdam.

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Florian Cajori.

University of California,
March, 1919.

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INTRODUCTION

THE contemplation of the various steps by which mankind has come into possession of the vast stock of mathematical knowledge can hardly fail to interest the mathematician. He takes pride in the fact that his science, more than any other, is an *exact* science, and that hardly anything ever done in mathematics has proved to be useless. The chemist smiles at the childish efforts of alchemists, but the mathematician finds the geometry of the Greeks and the arithmetic of the Hindus as useful and admirable as any research of to-day. He is pleased to notice that though, in course of its development, mathematics has had periods of slow growth, yet in the main it has been pre-eminently a *progressive* science.

The history of mathematics may be instructive as well as agreeable; it may not only remind us of what we have, but may also teach us how to increase our store. Says A. De Morgan, "The early history of the mind of men with regard to mathematics leads us to point out our own errors; and in this respect it is well to pay attention to the history of mathematics." It warns us against hasty conclusions; it points out the importance of a good notation upon the progress of the science; it discourages excessive specialisation on the part of investigators, by showing how apparently distinct branches have been found to possess unexpected connecting links; it saves the student from wasting time and energy upon problems which were, perhaps, solved long since; it discourages him from attacking an unsolved problem by the same method which has led other mathematicians to failure; it teaches that fortifications can be taken in other ways than by direct attack, that when repulsed from a direct assault it is well to reconnoitre and occupy the surrounding ground and to discover the secret paths by which the apparently unconquerable position can be taken.[1] The importance of this strategic rule may be emphasised by citing a case in which it has been violated. An untold amount of intellectual energy has been expended on the quadrature of the circle, yet no conquest has been made by direct assault. The circle-squarers have existed in crowds ever since the period of Archimedes. After innumerable failures to solve the problem at a time, even, when in-

[1] S. Günther, *Ziele und Resultate der neueren Mathematisch-historischen Forschung.* Erlangen, 1876.

vestigators possessed that most powerful tool, the differential calculus, persons versed in mathematics dropped the subject, while those who still persisted were completely ignorant of its history and generally misunderstood the conditions of the problem. "Our problem," says A. De Morgan, "is to square the circle with the *old allowance of means:* Euclid's postulates and nothing more. We cannot remember an instance in which a question to be solved by a *definite method* was tried by the best heads, and answered at last, *by that method*, after thousands of complete failures." But progress was made on this problem by approaching it from a different direction and by newly discovered paths. J. H. Lambert proved in 1761 that the ratio of the circumference of a circle to its diameter is irrational. Some years ago, F. Lindemann demonstrated that this ratio is also transcendental and that the quadrature of the circle, by means of the ruler and compasses only, is *impossible.* He thus showed by actual proof that which keen-minded mathematicians had long suspected; namely, that the great army of circle-squarers have, for two thousand years, been assaulting a fortification which is as indestructible as the firmament of heaven.

Another reason for the desirability of historical study is the value of historical knowledge to the teacher of mathematics. The interest which pupils take in their studies may be greatly increased if the solution of problems and the cold logic of geometrical demonstrations are interspersed with historical remarks and anecdotes. A class in arithmetic will be pleased to hear about the Babylonians and Hindus and their invention of the "Arabic notation"; they will marvel at the thousands of years which elapsed before people had even thought of introducing into the numeral notation that Columbus-egg—the zero; they will find it astounding that it should have taken so long to *invent* a notation which they themselves can now *learn* in a month. After the pupils have learned how to bisect a given angle, surprise them by telling of the many futile attempts which have been made to solve, by elementary geometry, the apparently very simple problem of the trisection of an angle. When they know how to construct a square whose area is double the area of a given square, tell them about the duplication of the cube, of its mythical origin—how the wrath of Apollo could be appeased only by the construction of a cubical altar double the given altar, and how mathematicians long wrestled with this problem. After the class have exhausted their energies on the theorem of the right triangle, tell them the legend about its discoverer—how Pythagoras, jubilant over his great accomplishment, sacrificed a hecatomb to the Muses who inspired him. When the value of mathematical training is called in question, quote the inscription over the entrance into the academy of Plato, the philosopher: "Let no one who is unacquainted with geometry enter here." Students in analytical geometry should know something of Descartes, and,

after taking up the differential and integral calculus, they should become familiar with the parts that Newton, Leibniz, and Lagrange played in creating that science. In his historical talk it is possible for the teacher to make it plain to the student that mathematics is not a dead science, but a living one in which steady progress is made.[1]

A similar point of view is taken by Henry S. White: [2] "The accepted truths of to-day, even the commonplace truths of any science, were the doubtful or the novel theories of yesterday. Some indeed of prime importance were long esteemed of slight importance and almost forgotten. The first effect of reading in the history of science is a naïve astonishment at the darkness of past centuries, but the ultimate effect is a fervent admiration for the progress achieved by former generations, for the triumphs of persistence and of genius. The easy credulity with which a young student supposes that of course every algebraic equation must have a root gives place finally to a delight in the slow conquest of the realm of imaginary numbers, and in the youthful genius of a Gauss who could demonstrate this once obscure fundamental proposition."

The history of mathematics is important also as a valuable contribution to the history of civilisation. Human progress is closely identified with scientific thought. Mathematical and physical researches are a reliable record of intellectual progress. The history of mathematics is one of the large windows through which the philosophic eye looks into past ages and traces the line of intellectual development.

[1] Cajori, F., *The Teaching and History of Mathematics in the United States.* Washington, 1890, p. 236.

[2] *Bull. Am. Math. Soc.*, Vol. 15, 1909, p. 325.

THE BABYLONIANS

THE fertile valley of the Euphrates and Tigris was one of the primeval seats of human society. Authentic history of the peoples inhabiting this region begins only with the foundation, in Chaldæa and Babylonia, of a united kingdom out of the previously disunited tribes. Much light has been thrown on their history by the discovery of the art of reading the *cuneiform* or wedge-shaped system of writing.

In the study of Babylonian mathematics we begin with the notation of numbers. A vertical wedge stood for 1, while the characters and signified 10 and 100 respectively. Grotefend believes the character for 10 originally to have been the picture of two hands, as held in prayer, the palms being pressed together, the fingers close to each other, but the thumbs thrust out. In the Babylonian notation two principles were employed—the additive and multiplicative. Numbers below 100 were expressed by symbols whose respective values had to be *added*. Thus, stood for 2, for 3, for 4, for 23, for 30. Here the symbols of higher order appear always to the left of those of lower order. In writing the hundreds, on the other hand, a *smaller* symbol was placed to the left of 100, and was, in that case, to be *multiplied* by 100. Thus, signified 10 times 100, or 1000. But this symbol for 1000 was itself taken for a new unit, which could take smaller coefficients to its left. Thus, denoted, not 20 times 100, but 10 times 1000. Some of the cuneiform numbers found on tablets in the ancient temple library at Nippur exceed a million; moreover, some of these Nippur tablets exhibit the *subtractive* principle (20–1), similar to that shown in the Roman notation, "XIX."

If, as is believed by most specialists, the early Sumerians were the inventors of the cuneiform writing, then they were, in all probability, also the inventors of the notation of numbers. Most surprising, in this connection, is the fact that Sumerian inscriptions disclose the use, not only of the above *decimal* system, but also of a *sexagesimal* one. The latter was used chiefly in constructing tables for weights and measures. It is full of historical interest. Its consequential development, both for integers and fractions, reveals a high degree of mathematical insight. We possess two Babylonian tablets which exhibit its use. One of them, probably written between 2300 and 1600 B. C., contains a table of square numbers up to 60^2. The numbers 1, 4, 9, 16, 25, 36, 49, are given as the squares of the first seven integers re-

spectively. We have next $1.4=8^2$, $1.21=9^2$, $1.40=10^2$, $2.1=11^2$, etc. This remains unintelligible, unless we assume the sexagesimal scale, which makes $1.4=60+4$, $1.21=60+21$, $2.1=2.60+1$. The second tablet records the magnitude of the illuminated portion of the moon's disc for every day from new to full moon, the whole disc being assumed to consist of 240 parts. The illuminated parts during the first five days are the series 5, 10, 20, 40, 1.20 (=80), which is a geometrical progression. From here on the series becomes an arithmetical progression, the numbers from the fifth to the fifteenth day being respectively 1.20, 1.36, 1.52, 2.8, 2.24, 2.40, 2.56, 3.12, 3.28, 3.44, 4. This table not only exhibits the use of the sexagesimal system, but also indicates the acquaintance of the Babylonians with progressions. Not to be overlooked is the fact that in the sexagesimal notation of integers the "principle of position" was employed. Thus, in 1.4 (=64), the 1 is made to stand for 60, the unit of the second order, by virtue of its position with respect to the 4. The introduction of this principle at so early a date is the more remarkable, because in the decimal notation it was not regularly introduced till about the ninth century after Christ. The principle of position, in its general and systematic application, requires a symbol for zero. We ask, Did the Babylonians possess one? Had they already taken the gigantic step of representing by a symbol the *absence* of units? Neither of the above tables answers this question, for they happen to contain no number in which there was occasion to use a zero. Babylonian records of many centuries later—of about 200 B. C.—give a symbol for zero which denoted the absence of a figure but apparently was not used in calculation. It consisted of two angular marks $\lessgtr$ one above the other, roughly resembling two dots, hastily written. About 130 A. D., Ptolemy in Alexandria used in his *Almagest* the Babylonian sexagesimal fractions, and also the omicron o to represent blanks in the sexagesimal numbers. This o was not used as a regular zero. It appears therefore that the Babylonians had the principle of local value, and also a symbol for zero, to indicate the absence of a figure, but did not use this zero in computation. Their sexagesimal fractions were introduced into India and with these fractions probably passed the principle of local value and the restricted use of the zero.

The sexagesimal system was used also in fractions. Thus, in the Babylonian inscriptions, $\frac{1}{2}$ and $\frac{1}{3}$ are designated by 30 and 20, the reader being expected, in his mind, to supply the word "sixtieths." The astronomer Hipparchus, the geometer Hypsicles and the astronomer Ptolemy borrowed the sexagesimal notation of fractions from the Babylonians and introduced it into Greece. From that time sexagesimal fractions held almost full sway in astronomical and mathematical calculations until the sixteenth century, when they finally yielded their place to the decimal fractions. It may be asked, What led to the invention of the sexagesimal system? Why was it that 60

parts were selected? To this we have no positive answer. *Ten* was chosen, in the decimal system, because it represents the number of fingers. But nothing of the human body could have suggested 60. Did the system have an astronomical origin? It was supposed that the early Babylonians reckoned the year at 360 days, that this led to the division of the circle into 360 degrees, each degree representing the daily amount of the supposed yearly revolution of the sun around the earth. Now they were, very probably, familiar with the fact that the radius can be applied to its circumference as a chord 6 times, and that each of these chords subtends an arc measuring exactly 60 degrees. Fixing their attention upon these degrees, the division into 60 parts may have suggested itself to them. Thus, when greater precision necessitated a subdivision of the degree, it was partitioned into 60 minutes. In this way the sexagesimal notation was at one time supposed to have originated. But it now appears that the Babylonians very early knew that the year exceeded 360 days. Moreover, it is highly improbable that a higher unit of 360 was chosen first, and a lower unit of 60 afterward. The normal development of a number system is from lower to higher units. Another guess is that the sexagesimal system arose as a mixture of two earlier systems of the bases 6 and 10.[1] Certain it is that the sexagesimal system became closely interwoven with astronomical and geometrical science. The division of the day into 24 hours, and of the hour into minutes and seconds on the scale of 60, is attributed to the Babylonians. There is strong evidence for the belief that they had also a division of the day into 60 hours. The employment of a sexagesimal division in numeral notation, in fractions, in angular as well as in time measurement, indicated a beautiful harmony which was not disturbed for thousands of years until Hindu and Arabic astronomers began to use sines and cosines in place of parts of chords, thereby forcing the right angle to the front as a new angular unit, which, for consistency, should have been subdivided sexagesimally, but was not actually so divided.

It appears that the people in the Tigro-Euphrates basin had made very creditable advance in arithmetic. Their knowledge of arithmetical and geometrical progressions has already been alluded to. Iamblichus attributes to them also a knowledge of proportion, and even the invention of the so-called *musical* proportion. Though we

[1] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 1. Bd., 3. Aufl., Leipzig, 1907, p. 37. This work has appeared in four large volumes and carries the history down to 1799. The fourth volume (1908) was written with the coöperation of nine scholars from Germany, Italy, Russia and the United States. *Moritz Cantor* (1829–) ranks as the foremost general writer of the nineteenth century on the history of mathematics. Born in Mannheim, a student at Heidelberg, at Göttingen under Gauss and Weber, at Berlin under Dirichlet, he lectured at Heidelberg where in 1877 he became ordinary honorary professor. His first historical article was brought out in 1856, but not until 1880 did the first volume of his well-known history appear.

possess no conclusive proof, we have nevertheless reason to believe that in practical calculation they used the *abacus*. Among the races of middle Asia, even as far as China, the abacus is as old as fable. Now, Babylon was once a great commercial centre,—the metropolis of many nations,—and it is, therefore, not unreasonable to suppose that her merchants employed this most improved aid to calculation.

In 1889 H. V. Hilprecht began to make excavations at Nuffar (the ancient Nippur) and found brick tablets containing multiplication and division tables, tables of squares and square roots, a geometric progression and a few computations. He published an account of his findings in 1906.[1]

The divisions in one tablet contain results like these: "60^4 divided by $2 = 6{,}480{,}000$ each," "60^4 divided by $3 = 4{,}320{,}000$ each," and so on, using the divisors 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18. The very first division on the tablet is interpreted to be "60^4 divided by $1\frac{1}{2} = 8{,}640{,}000$." This strange appearance of $\frac{3}{2}$ as a divisor is difficult to explain. Perhaps there is here a use of $\frac{3}{2}$ corresponding to the Egyptian use of $\frac{2}{3}$ as found in the Ahmes papyrus at a, perhaps, contemporaneous period. It is noteworthy that $60^4 = 12{,}960{,}000$, which Hilprecht found in the Nippur brick text-books, is nothing less than the mystic "Platonic number," the "lord of better and worse births," mentioned in Plato's *Republic*. Most probably, Plato received the number from the Pythagoreans, and the Pythagoreans from the Babylonians.[2]

In geometry the Babylonians accomplished little. Besides the division of the circumference into 6 parts by its radius, and into 360 degrees, they had some knowledge of geometrical figures, such as the triangle and quadrangle, which they used in their auguries. Like the Hebrews (1 Kin. 7:23), they took $\pi = 3$. Of geometrical demonstrations there is, of course, no trace. "As a rule, in the Oriental mind the intuitive powers eclipse the severely rational and logical."

Hilprecht concluded from his studies that the Babylonians possessed the rules for finding the areas of squares, rectangles, right triangles, and trapezoids.

The astronomy of the Babylonians has attracted much attention. They worshipped the heavenly bodies from the earliest historic times. When Alexander the Great, after the battle of Arbela (331 B. C.), took possession of Babylon, Callisthenes found there on burned brick astronomical records reaching back as far as 2234 B. C. Porphyrius says that these were sent to Aristotle. Ptolemy, the Alexandrian astronomer, possessed a Babylonian record of eclipses going back to 747 B. C.

[1] *Mathematical, Metrological and Chronological Tablets from the Temple Library of Nippur*, by H. V. Hilprecht. Vol. XX, part I, Series A, Cuneiform Texts, published by the Babylonian Expedition of the University of Pennsylvania, 1906. Consult also D. E. Smith in *Bull. Am. Math. Soc.*, Vol. 13, 1907, p. 392.

[2] On the "Platonic number" consult P. Tannery in *Revue philosophique*, Vol. I, 1876, p. 170; Vol. XIII, 1881, p. 210; Vol. XV, 1883, p. 573. Also G. Loria in *Le scienze esatte nell'antica grecia*, 2 Ed., Milano, 1914, Appendice.

Epping and Strassmaier [1] have thrown considerable light on Babylonian chronology and astronomy by explaining two calendars of the years 123 B. C. and 111 B. C., taken from cuneiform tablets coming, presumably, from an old observatory. These scholars have succeeded in giving an account of the Babylonian calculation of the new and full moon, and have identified by calculations the Babylonian names of the planets, and of the twelve zodiacal signs and twenty-eight normal stars which correspond to some extent with the twenty-eight *nakshatras* of the Hindus. We append part of an Assyrian astronomical report, as translated by Oppert:—

"To the King, my lord, thy faithful servant, Mar-Istar."

". . . On the first day, as the new moon's day of the month Thammuz declined, the moon was again visible over the planet Mercury, as I had already predicted to my master the King. I erred not."

[1] Epping, J., *Astronomisches aus Babylon. Unter Mitwirkung von* P. J. R. Strassmaier. Freiburg, 1889.

THE EGYPTIANS

Though there is difference of opinion regarding the antiquity of Egyptian civilisation, yet all authorities agree in the statement that, however far back they go, they find no uncivilised state of society. "Menes, the first king, changes the course of the Nile, makes a great reservoir, and builds the temple of Phthah at Memphis." The Egyptians built the pyramids at a very early period. Surely a people engaging in enterprises of such magnitude must have known something of mathematics—at least of practical mathematics.

All Greek writers are unanimous in ascribing, without envy, to Egypt the priority of invention in the mathematical sciences. Plato in *Phœdrus* says: "At the Egyptian city of Naucratis there was a famous old god whose name was Theuth; the bird which is called the Ibis was sacred to him, and he was the inventor of many arts, such as arithmetic and calculation and geometry and astronomy and draughts and dice, but his great discovery was the use of letters."

Aristotle says that mathematics had its birth in Egypt, because there the priestly class had the leisure needful for the study of it. Geometry, in particular, is said by Herodotus, Diodorus, Diogenes Laertius, Iamblichus, and other ancient writers to have originated in Egypt.[1] In Herodotus we find this (II. c. 109): "They said also that this king [Sesostris] divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levied yearly. But every one from whose part the river tore away anything, had to go to him and notify what had happened; he then sent the overseers, who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax imposed. In this way, it appears to me, geometry originated, which passed thence to Hellas."

We abstain from introducing additional Greek opinion regarding Egyptian mathematics, or from indulging in wild conjectures. We rest our account on documentary evidence. A hieratic papyrus, included in the Rhind collection of the British Museum, was deciphered by Eisenlohr in 1877, and found to be a mathematical manual containing problems in arithmetic and geometry. It was written by **Ahmes** some time before 1700 B. C., and was founded on an older work believed by Birch to date back as far as 3400 B. C.! This curious

[1] C. A. Bretschneider *Die Geometrie und die Geometer vor Euklides*. Leipzig, 1870, pp. 6–8. Carl Anton Bretschneider (1808–1878) was professor at the Realgymnasium at Gotha in Thuringia.

papyrus—the most ancient mathematical handbook known to us—puts us at once in contact with the mathematical thought in Egypt of three or five thousand years ago. It is entitled "Directions for obtaining the Knowledge of all Dark Things." We see from it that the Egyptians cared but little for theoretical results. Theorems are not found in it at all. It contains "hardly any general rules of procedure, but chiefly mere statements of results intended possibly to be explained by a teacher to his pupils." [1] In geometry the forte of the Egyptians lay in making constructions and determining areas. The area of an isosceles triangle, of which the sides measure 10 *khets* (a unit of length equal to 16.6 *m.* by one guess and about thrice that amount by another guess [2]) and the base 4 *khets*, was erroneously given as 20 square *khets*, or half the product of the base by one side. The area of an isosceles trapezoid is found, similarly, by multiplying half the sum of the parallel sides by one of the non-parallel sides. The area of a circle is found by deducting from the diameter $\frac{1}{9}$ of its length and squaring the remainder. Here π is taken $=(\frac{16}{9})^2=3.1604...$, a very fair approximation. The papyrus explains also such problems as these,—To mark out in the field a right triangle whose sides are 10 and 4 units; or a trapezoid whose parallel sides are 6 and 4, and the non-parallel sides each 20 units.

Some problems in this papyrus seem to imply a rudimentary knowledge of proportion.

The base-lines of the pyramids run north and south, and east and west, but probably only the lines running north and south were determined by astronomical observations. This, coupled with the fact that the word *harpedonaptæ*, applied to Egyptian geometers, means "rope-stretchers," would point to the conclusion that the Egyptian, like the Indian and Chinese geometers, constructed a right triangle upon a given line, by stretching around three pegs a rope consisting of three parts in the ratios $3:4:5$, and thus forming a right triangle.[3] If this explanation is correct, then the Egyptians were familiar, 2000 years B. C., with the well-known property of the right triangle, for the special case at least when the sides are in the ratio $3:4:5$.

On the walls of the celebrated temple of Horus at Edfu have been found hieroglyphics, written about 100 B. C., which enumerate the pieces of land owned by the priesthood, and give their areas. The area of any quadrilateral, however irregular, is there found by the formula $\frac{a+b}{2}\cdot\frac{c+d}{2}$. Thus, for a quadrangle whose opposite sides are 5 and 8, 20 and 15, is given the area $113\frac{1}{2}\ \frac{1}{4}$.[4] The incorrect for-

[1] James Gow, *A Short History of Greek Mathematics*. Cambridge, 1884, p. 16.

[2] A. Eisenlohr, *Ein mathematisches Handbuch der alten Aegypter*, 2. Ausgabe, Leipzig, 1897, p. 103; F. L. Griffith in *Proceedings of the Society of Biblical Archæology*, 1891, 1894.

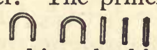
[3] M. Cantor, *op. cit.* Vol. I, 3. Aufl., 1907, p. 105.

[4] H. Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, Leipzig, 1874, p. 86.

mulæ of Ahmes of 3000 years B. C. yield generally closer approximations than those of the Edfu inscriptions, written 200 years after Euclid!

The fact that the geometry of the Egyptians consists chiefly of constructions, goes far to explain certain of its great defects. The Egyptians failed in two essential points without which a *science* of geometry, in the true sense of the word, cannot exist. In the first place, they failed to construct a rigorously logical system of geometry, resting upon a few axioms and postulates. A great many of their rules, especially those in solid geometry, had probably not been proved at all, but were known to be true merely from observation or as matters of fact. The second great defect was their inability to bring the numerous special cases under a more general view, and thereby to arrive at broader and more fundamental theorems. Some of the simplest geometrical truths were divided into numberless special cases of which each was supposed to require separate treatment.

Some particulars about Egyptian geometry can be mentioned more advantageously in connection with the early Greek mathematicians who came to the Egyptian priests for instruction.

An insight into Egyptian methods of numeration was obtained through the ingenious deciphering of the hieroglyphics by Champollion, Young, and their successors. The symbols used were the following:  for 1,  for 10,  for 100,  for 1000,  for 10,000,  for 100,000,  for 1,000,000,  for 10,000,000.[1] The symbol for 1 represents a vertical staff; that for 10,000 a pointing finger; that for 100,000 a burbot; that for 1,000,000, a man in astonishment. The significance of the remaining symbols is very doubtful. The writing of numbers with these hieroglyphics was very cumbrous. The unit symbol of each order was repeated as many times as there were units in that order. The principle employed was the *additive*. Thus, 23 was written 

Besides the hieroglyphics, Egypt possesses the *hieratic* and *demotic* writings, but for want of space we pass them by.

Herodotus makes an important statement concerning the mode of computing among the Egyptians. He says that they "*calculate with pebbles* by moving the hand from right to left, while the Hellenes move it from left to right." Herein we recognise again that *instrumental* method of figuring so extensively used by peoples of antiquity. The Egyptians used the decimal scale. Since, in figuring, they moved their hands horizontally, it seems probable that they used ciphering-boards with vertical columns. In each column there must have been not more than nine pebbles, for ten pebbles would be equal to one pebble in the column next to the left.

[1] M. Cantor, *op. cit.* Vol. I, 3. Aufl., 1907, p. 82.

The *Ahmes papyrus* contains interesting information on the way in which the Egyptians employed fractions. Their methods of operation were, of course, radically different from ours. Fractions were a subject of very great difficulty with the ancients. Simultaneous changes in both numerator and denominator were usually avoided. In manipulating fractions the Babylonians kept the denominators (60) constant. The Romans likewise kept them constant, but equal to 12. The Egyptians and Greeks, on the other hand, kept the numerators constant, and dealt with variable denominators. Ahmes used the term "fraction" in a restricted sense, for he applied it only to *unit-fractions*, or fractions having unity for the numerator. It was designated by writing the denominator and then placing over it a dot. Fractional values which could not be expressed by any one unit-fraction were expressed as the *sum* of two or more of them. Thus, he wrote $\frac{1}{3}\ \frac{1}{15}$ in place of $\frac{2}{5}$. While Ahmes knows $\frac{2}{3}$ to be equal to $\frac{1}{2}\ \frac{1}{6}$, he curiously allows $\frac{2}{3}$ to appear often among the unit-fractions and adopts a special symbol for it. The first important problem naturally arising was, how to represent any fractional value as the sum of unit-fractions. This was solved by aid of a table, given in the papyrus, in which all fractions of the form $\frac{2}{2n+1}$ (where n designates successively all the numbers up to 49) are reduced to the sum of unit-fractions. Thus, $\frac{2}{7}=\frac{1}{4}\ \frac{1}{28}$; $\frac{2}{99}=\frac{1}{66}\ \frac{1}{198}$. When, by whom, and how this table was calculated, we do not know. Probably it was compiled empirically at different times, by different persons. It will be seen that by repeated application of this table, a fraction whose numerator exceeds two can be expressed in the desired form, provided that there is a fraction in the table having the same denominator that *it* has. Take, for example, the problem, to divide 5 by 21. In the first place, $5=1+2+2$. From the table we get $\frac{2}{21}=\frac{1}{14}\ \frac{1}{42}$. Then $\frac{5}{21}=\frac{1}{21}+(\frac{1}{14}\ \frac{1}{42})+(\frac{1}{14}\ \frac{1}{42})=\frac{1}{21}+(\frac{2}{14}\ \frac{2}{42})=\frac{1}{21}\ \frac{1}{7}\ \frac{1}{21}=\frac{1}{7}\ \frac{2}{21}=\frac{1}{7}\ \frac{1}{14}\ \frac{1}{42}$. The papyrus contains problems in which it is required that fractions be raised by addition or multiplication to given whole numbers or to other fractions. For example, it is required to increase $\frac{1}{4}\ \frac{1}{8}\ \frac{1}{10}\ \frac{1}{30}\ \frac{1}{45}$ to 1. The common denominator taken appears to be 45, for the numbers are stated as $11\frac{1}{4}$, $5\frac{1}{2}\ \frac{1}{8}$, $4\frac{1}{2}$, $1\frac{1}{2}$, 1. The sum of these is $23\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}$ forty-fifths. Add to this $\frac{1}{9}\ \frac{1}{40}$, and the sum is $\frac{2}{3}$. Add $\frac{1}{3}$, and we have 1. Hence the quantity to be added to the given fraction is $\frac{1}{3}\ \frac{1}{9}\ \frac{1}{40}$.

Ahmes gives the following example involving an *arithmetical progression:* "Divide 100 loaves among 5 persons; $\frac{1}{7}$ of what the first three get is what the last two get. What is the difference?" Ahmes gives the solution: "Make the difference $5\frac{1}{2}$; 23, $17\frac{1}{2}$, 12, $6\frac{1}{2}$, 1. Multiply by $1\frac{2}{3}$; $38\frac{1}{3}$, $29\frac{1}{6}$, 20, $10\frac{2}{3}\ \frac{1}{6}$, $1\frac{2}{3}$." How did Ahmes come upon

$5\frac{1}{2}$? Perhaps thus:[1] Let a and $-d$ be the first term and the difference in the required arithmetical progression, then $\frac{1}{7}[a+(a-d)+(a-2d)]=(a-3d)+(a-4d)$, whence $d=5\frac{1}{2}(a-4d)$, *i. e.* the difference d is $5\frac{1}{2}$ times the last term. *Assuming* the last term 1, he gets his first progression. The sum is 60, but should be 100; hence multiply by $1\frac{2}{3}$, for $60\times1\frac{2}{3}=100$. We have here a method of solution which appears again later among the Hindus, Arabs and modern Europeans—the famous method of *false position.*

Ahmes speaks of a ladder consisting of the numbers 7, 49, 343, 2401, 16807. Adjacent to these powers of 7 are the words *picture, cat, mouse, barley, measure.* What is the meaning of these mysterious data? Upon the consideration of the problem given by Leonardo of Pisa in his *Liber abaci*, 3000 years later: "7 old women go to Rome, each woman has 7 mules, each mule carries 7 sacks, etc.", Moritz Cantor offers the following solution to the Ahmes riddle: 7 persons have each 7 cats, each cat eats 7 mice, each mouse eats 7 ears of barley, from each ear 7 measures of corn may grow. How many persons, cats, mice, ears of barley, and measures of corn, altogether? Ahmes gives 19607 as the sum of the geometric progression. Thus, the Ahmes papyrus discloses a knowledge of both arithmetical and geometrical progression.

Ahmes proceeds to the solution of equations of one unknown quantity. The unknown quantity is called 'hau' or heap. Thus the problem, "heap, its $\frac{1}{7}$, its whole, it makes 19," *i. e.* $\frac{x}{7}+x=19$. In this case, the solution is as follows: $\frac{8x}{7}=19$; $\frac{x}{7}=2\frac{1}{4}\,\frac{1}{8}$; $x=16\frac{1}{2}\,\frac{1}{8}$. But in other problems, the solutions are effected by various other methods. It thus appears that the beginnings of algebra are as ancient as those of geometry.

That the period of Ahmes was a flowering time for Egyptian mathematics appears from the fact that there exist other papyri (more recently discovered) of the same period. They were found at Kahun, south of the pyramid of Illahun. These documents bear close resemblance to Ahmes. They contain, moreover, examples of quadratic equations, the earliest of which we have a record. One of them is:[2] A given surface of, say, 100 units of area, shall be represented as the sum of two squares, whose sides are to each other as $1:\frac{3}{4}$. In modern symbols, the problem is, to find x and y, such that $x^2+y^2=100$ and $x:y=1:\frac{3}{4}$. The solution rests upon the method of false position. Try $x=1$ and $y=\frac{3}{4}$, then $x^2+y^2=\frac{25}{16}$ and $\sqrt{\frac{25}{16}}=\frac{5}{4}$. But $\sqrt{100}=10$ and $10\div\frac{5}{4}=8$. The rest of the solution cannot be made

[1] M. Cantor, *op. cit.*, Vol. I, 3. Aufl., 1907, p. 78.
[2] Cantor, *op. cit.* Vol. I, 1907, pp. 95, 96.

out, but probably was $x=8\times 1$, $y=8\times\frac{3}{4}=6$. This solution leads to the relation $6^2+8^2=10^2$. The symbol ┏ was used to designate square root.

In some ways similar to the Ahmes papyrus is also the Akhmim papyrus,[1] written over 2000 years later at Akhmim, a city on the Nile in Upper Egypt. It is in Greek and is supposed to have been written at some time between 500 and 800, A. D. It contains, besides arithmetical examples, a table for finding "unit-fractions," like that of Ahmes. Unlike Ahmes, it tells how the table was constructed. The rule, expressed in modern symbols, is as follows: $\frac{z}{p\cdot q}=\frac{1}{q\frac{p+q}{z}}+\frac{1}{p\frac{p+q}{z}}$ For $z=2$, this formula reproduces part of the table in Ahmes.

The principal defect of Egyptian arithmetic was the lack of a simple, comprehensive symbolism—a defect which not even the Greeks were able to remove.

The Ahmes papyrus and the other papyri of the same period represent the most advanced attainments of the Egyptians in arithmetic and geometry. It is remarkable that they should have reached so great proficiency in mathematics at so remote a period of antiquity. But strange, indeed, is the fact that, during the next two thousand years, they should have made no progress whatsoever in it. The conclusion forces itself upon us, that they resemble the Chinese in the *stationary character*, not only of their government, but also of their learning. All the knowledge of geometry which they possessed when Greek scholars visited them, six centuries B. C., was doubtless known to them two thousand years earlier, when they built those stupendous and gigantic structures—the pyramids.

[1] J. Baillet, "Le papyrus mathématique d'Akhmim," *Mémoires publiés par les membres de la mission archéologique française au Caire*, T. IX, 1r fascicule, Paris, 1892, pp. 1–88. See also Cantor, *op. cit.* Vol. I, 1907, pp. 67, 504.

THE GREEKS

Greek Geometry

About the seventh century B. C. an active commercial intercourse sprang up between Greece and Egypt. Naturally there arose an interchange of ideas as well as of merchandise. Greeks, thirsting for knowledge, sought the Egyptian priests for instruction. Thales, Pythagoras, Œnopides, Plato, Democritus, Eudoxus, all visited the land of the pyramids. Egyptian ideas were thus transplanted across the sea and there stimulated Greek thought, directed it into new lines, and gave to it a basis to work upon. Greek culture, therefore, is not primitive. Not only in mathematics, but also in mythology and art, Hellas owes a debt to older countries. To Egypt Greece is indebted, among other things, for its elementary geometry. But this does not lessen our admiration for the Greek mind. From the moment that Hellenic philosophers applied themselves to the study of Egyptian geometry, this science assumed a radically different aspect. "Whatever we Greeks receive, we improve and perfect," says Plato. The Egyptians carried geometry no further than was absolutely necessary for their practical wants. The Greeks, on the other hand, had within them a strong speculative tendency. They felt a craving to discover the reasons for things. They found pleasure in the contemplation of *ideal* relations, and loved science *as* science.

Our sources of information on the history of Greek geometry before Euclid consist merely of scattered notices in ancient writers. The early mathematicians, Thales and Pythagoras, left behind no written records of their discoveries. A full history of Greek geometry and astronomy during this period, written by Eudemus, a pupil of Aristotle, has been lost. It was well known to Proclus, who, in his commentaries on Euclid, gives a brief account of it. This abstract constitutes our most reliable information. We shall quote it frequently under the name of *Eudemian Summary.*

The Ionic School

To **Thales** (640–546 B. C.), of Miletus, one of the "seven wise men," and the founder of the Ionic school, falls the honor of having introduced the study of geometry into Greece. During middle life he engaged in commercial pursuits, which took him to Egypt. He is said to have resided there, and to have studied the physical sciences and mathematics with the Egyptian priests. Plutarch declares that Thales soon excelled his masters, and amazed King Amasis by measur-

ing the heights of the pyramids from their shadows. According to Plutarch, this was done by considering that the shadow cast by a vertical staff of known length bears the same ratio to the shadow of the pyramid as the height of the staff bears to the height of the pyramid. This solution presupposes a knowledge of proportion, and the Ahmes papyrus actually shows that the rudiments of proportion were known to the Egyptians. According to Diogenes Laertius, the pyramids were measured by Thales in a different way; viz. by finding the length of the shadow of the pyramid at the moment when the shadow of a staff was equal to its own length. Probably both methods were used.

The *Eudemian Summary* ascribes to Thales the invention of the theorems on the equality of vertical angles, the equality of the angles at the base of an isosceles triangle, the bisection of a circle by any diameter, and the congruence of two triangles having a side and the two adjacent angles equal respectively. The last theorem, combined (we have reason to suspect) with the theorem on similar triangles, he applied to the measurement of the distances of ships from the shore. Thus Thales was the first to apply theoretical geometry to practical uses. The theorem that all angles inscribed in a semicircle are right angles is attributed by some ancient writers to Thales, by others to Pythagoras. Thales was doubtless familiar with other theorems, not recorded by the ancients. It has been inferred that he knew the sum of the three angles of a triangle to be equal to two right angles, and the sides of equiangular triangles to be proportional.[1] The Egyptians must have made use of the above theorems on the straight line, in some of their constructions found in the Ahmes papyrus, but it was left for the Greek philosopher to give these truths, which others saw, but did not formulate into words, an explicit, abstract expression, and to put into scientific language and subject to proof that which others merely felt to be true. Thales may be said to have created the geometry of lines, essentially abstract in its character, while the Egyptians studied only the geometry of surfaces and the rudiments of solid geometry, empirical in their character.[2]

With Thales begins also the study of scientific astronomy. He acquired great celebrity by the prediction of a solar eclipse in 585 B.C. Whether he predicted the day of the occurrence, or simply the year, is not known. It is told of him that while contemplating the stars during an evening walk, he fell into a ditch. The good old woman attending him exclaimed, "How canst thou know what is doing in the heavens, when thou seest not what is at thy feet?"

The two most prominent pupils of Thales were **Anaximander** (b. 611

[1] G. J. Allman, *Greek Geometry from Thales to Euclid*. Dublin, 1889, p. 10. George Johnston Allman (1824–1904) was professor of mathematics at Queen's College, Galway, Ireland.

[2] G. J. Allman, *op. cit.*, p. 15.

B. C.) and **Anaximenes** (b. 570 B. C.). They studied chiefly astronomy and physical philosophy. Of **Anaxagoras** (500–428 B. C.), a pupil of Anaximenes, and the last philosopher of the Ionic school, we know little, except that, while in prison, he passed his time attempting to square the circle. This is the first time, in the history of mathematics, that we find mention of the famous problem of the quadrature of the circle, that rock upon which so many reputations have been destroyed. It turns upon the determination of the exact value of π. Approximations to π had been made by the Chinese, Babylonians, Hebrews, and Egyptians. But the invention of a method to find its *exact* value, is the knotty problem which has engaged the attention of many minds from the time of Anaxagoras down to our own. Anaxagoras did not offer any solution of it, and seems to have luckily escaped paralogisms. The problem soon attracted popular attention, as appears from the reference to it made in 414 B. C. by the comic poet Aristophanes in his play, the "Birds." [1]

About the time of Anaxagoras, but isolated from the Ionic school, flourished **Œnopides** of Chios. Proclus ascribes to him the solution of the following problems: From a point without, to draw a perpendicular to a given line, and to draw an angle on a line equal to a given angle. That a man could gain a reputation by solving problems so elementary as these, indicates that geometry was still in its infancy, and that the Greeks had not yet gotten far beyond the Egyptian constructions.

The Ionic school lasted over one hundred years. The progress of mathematics during that period was slow, as compared with its growth in a later epoch of Greek history. A new impetus to its progress was given by Pythagoras.

The School of Pythagoras

Pythagoras (580?–500? B. C.) was one of those figures which impressed the imagination of succeeding times to such an extent that their real histories have become difficult to be discerned through the mythical haze that envelops them. The following account of Pythagoras excludes the most doubtful statements. He was a native of Samos, and was drawn by the fame of Pherecydes to the island of Syros. He then visited the ancient Thales, who incited him to study in Egypt. He sojourned in Egypt many years, and may have visited Babylon. On his return to Samos, he found it under the tyranny of Polycrates. Failing in an attempt to found a school there, he quitted home again and, following the current of civilisation, removed to Magna Græcia in South Italy. He settled at Croton, and founded the famous Pythagorean school. This was not merely an academy for

[1] F. Rudio in *Bibliotheca mathematica*, 3 S., Vol. 8, 1907–8, pp. 13–22.

the teaching of philosophy, mathematics, and natural science, but it was a brotherhood, the members of which were united for life. This brotherhood had observances approaching masonic peculiarity. They were forbidden to divulge the discoveries and doctrines of their school. Hence we are obliged to speak of the Pythagoreans as a body, and find it difficult to determine to whom each particular discovery is to be ascribed. The Pythagoreans themselves were in the habit of referring every discovery back to the great founder of the sect.

This school grew rapidly and gained considerable political ascendency. But the mystic and secret observances, introduced in imitation of Egyptian usages, and the aristocratic tendencies of the school, caused it to become an object of suspicion. The democratic party in Lower Italy revolted and destroyed the buildings of the Pythagorean school. Pythagoras fled to Tarentum and thence to Metapontum, where he was murdered.

Pythagoras has left behind no mathematical treatises, and our sources of information are rather scanty. Certain it is that, in the Pythagorean school, mathematics was the principal study. Pythagoras raised mathematics to the rank of a science. Arithmetic was courted by him as fervently as geometry. In fact, arithmetic is the foundation of his philosophic system.

The *Eudemian Summary* says that "Pythagoras changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom, and investigated its theorems in an immaterial and intellectual manner." His geometry was connected closely with his arithmetic. He was especially fond of those geometrical relations which admitted of arithmetical expression.

Like Egyptian geometry, the geometry of the Pythagoreans is much concerned with areas. To Pythagoras is ascribed the important theorem that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides. He had probably learned from the Egyptians the truth of the theorem in the special case when the sides are 3, 4, 5, respectively. The story goes, that Pythagoras was so jubilant over this discovery that he sacrificed a hecatomb. Its authenticity is doubted, because the Pythagoreans believed in the transmigration of the soul and opposed the shedding of blood. In the later traditions of the Neo-Pythagoreans this objection is removed by replacing this bloody sacrifice by that of "an ox made of flour!" The proof of the law of three squares, given in Euclid's *Elements*, I. 47, is due to Euclid himself, and not to the Pythagoreans. What the Pythagorean method of proof was has been a favorite topic for conjecture.

The theorem on the sum of the three angles of a triangle, presumably known to Thales, was proved by the Pythagoreans after the manner of Euclid. They demonstrated also that the plane about a point is completely filled by six equilateral triangles, four squares, or

three regular hexagons, so that it is possible to divide up a plane into figures of either kind.

From the equilateral triangle and the square arise the solids, namely, the tetraedron, octaedron, icosaedron, and the cube. These solids were, in all probability, known to the Egyptians, excepting, perhaps, the icosaedron. In Pythagorean philosophy, they represent respectively the four elements of the physical world; namely, fire, air, water, and earth. Later another regular solid was discovered, namely, the dodecaedron, which, in absence of a fifth element, was made to represent the universe itself. Iamblichus states that Hippasus, a Pythagorean, perished in the sea, because he boasted that he first divulged "the sphere with the twelve pentagons." The same story of death at sea is told of a Pythagorean who disclosed the theory of irrationals. The star-shaped pentagram was used as a symbol of recognition by the Pythagoreans, and was called by them Health.

Pythagoras called the sphere the most beautiful of all solids, and the circle the most beautiful of all plane figures. The treatment of the subjects of proportion and of irrational quantities by him and his school will be taken up under the head of arithmetic.

According to Eudemus, the Pythagoreans invented the problems concerning the application of areas, including the cases of defect and excess, as in Euclid, VI. 28, 29.

They were also familiar with the construction of a polygon equal in area to a given polygon and similar to another given polygon. This problem depends upon several important and somewhat advanced theorems, and testifies to the fact that the Pythagoreans made no mean progress in geometry.

Of the theorems generally ascribed to the Italian school, some cannot be attributed to Pythagoras himself, nor to his earliest successors. The progress from empirical to reasoned solutions must, of necessity, have been slow. It is worth noticing that on the circle no theorem of any importance was discovered by this school.

Though politics broke up the Pythagorean fraternity, yet the school continued to exist at least two centuries longer. Among the later Pythagoreans, Philolaus and Archytas are the most prominent. **Philolaus** wrote a book on the Pythagorean doctrines. By him were first given to the world the teachings of the Italian school, which had been kept secret for a whole century. The brilliant **Archytas** (428–347 B. C.) of Tarentum, known as a great statesman and general, and universally admired for his virtues, was the only great geometer among the Greeks when Plato opened his school. Archytas was the first to apply geometry to mechanics and to treat the latter subject methodically. He also found a very ingenious mechanical solution to the problem of the duplication of the cube. His solution involves clear notions on the generation of cones and cylinders. This problem reduces itself to finding two mean proportionals between two given

lines. These mean proportionals were obtained by Archytas from the section of a half-cylinder. The doctrine of proportion was advanced through him.

There is every reason to believe that the later Pythagoreans exercised a strong influence on the study and development of mathematics at Athens. The Sophists acquired geometry from Pythagorean sources. Plato bought the works of Philolaus, and had a warm friend in Archytas.

The Sophist School

After the defeat of the Persians under Xerxes at the battle of Salamis, 480 B. C., a league was formed among the Greeks to preserve the freedom of the now liberated Greek cities on the islands and coast of the Ægæan Sea. Of this league Athens soon became leader and dictator. She caused the separate treasury of the league to be merged into that of Athens, and then spent the money of her allies for her own aggrandisement. Athens was also a great commercial centre. Thus she became the richest and most beautiful city of antiquity. All menial work was performed by slaves. The citizen of Athens was well-to-do and enjoyed a large amount of leisure. The government being purely democratic, every citizen was a politician. To make his influence felt among his fellow-men he must, first of all, be educated. Thus there arose a demand for teachers. The supply came principally from Sicily, where Pythagorean doctrines had spread. These teachers were called *Sophists*, or "wise men." Unlike the Pythagoreans, they accepted pay for their teaching. Although rhetoric was the principal feature of their instruction, they also taught geometry, astronomy, and philosophy. Athens soon became the headquarters of Grecian men of letters, and of mathematicians in particular. The home of mathematics among the Greeks was first in the Ionian Islands, then in Lower Italy, and during the time now under consideration, at Athens.

The geometry of the circle, which had been entirely neglected by the Pythagoreans, was taken up by the Sophists. Nearly all their discoveries were made in connection with their innumerable attempts to solve the following three famous problems:—

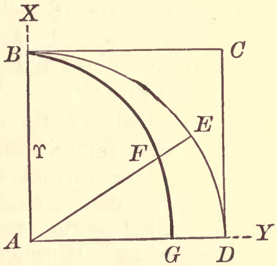
(1) To trisect an arc or an angle.

(2) To "double the cube," *i. e.*, to find a cube whose *volume* is double that of a given cube.

(3) To "square the circle," *i. e.* to find a square or some other rectilinear figure exactly equal in area to a given circle.

These problems have probably been the subject of more discussion and research than any other problems in mathematics. The bisection of an angle was one of the easiest problems in geometry. The trisection of an angle, on the other hand, presented unexpected difficulties. A right angle had been divided into three equal parts by the Pytha-

goreans. But the general construction, though easy in appearance, cannot be effected by the aid only of ruler and compasses. Among the first to wrestle with it was **Hippias of Elis,** a contemporary of Socrates, and born about 460 B. C. Unable to reach a solution by ruler and compasses only, he and other Greek geometers resorted to the use of other means. Proclus mentions a man, Hippias, presumably Hippias of Elis, as the inventor of a transcendental curve which served to divide an angle not only into three, but into any number of equal parts. This same curve was used later by Dinostratus and others for the quadrature of the circle. On this account it is called the *quadratrix.* The curve may be described thus: The side AB of the square shown in the figure turns uniformly about A, the point B moving along the circular arc BED. In the same time, the side BC moves parallel to itself and uniformly from the position of BC to that of AD. The locus of intersection of AB and BC, when thus moving, is the quadratrix BFG. Its equation we now write $y = x \cot \frac{\pi x}{2r}$. The ancients considered only the part of the curve that lies inside the quadrant of the circle; they did not know that $x = \pm 2r$ are asymptotes, nor that there is an infinite number of branches. According to Pappus, Dinostratus effected the quadrature by establishing the theorem that BED:AD = AD:AG.



The Pythagoreans had shown that the diagonal of a square is the side of another square having double the area of the original one. This probably suggested the problem of the duplication of the cube, *i. e.*, to find the edge of a cube having double the volume of a given cube. Eratosthenes ascribes to this problem a different origin. The Delians were once suffering from a pestilence and were ordered by the oracle to double a certain cubical altar. Thoughtless workmen simply constructed a cube with edges twice as long, but brainless work like that did not pacify the gods. The error being discovered, Plato was consulted on the matter. He and his disciples searched eagerly for a solution to this "Delian Problem." An important contribution to this problem was made by **Hippocrates of Chios** (about 430 B. C.). He was a talented mathematician but, having been defrauded of his property, he was pronounced slow and stupid. It is also said of him that he was the first to accept pay for the teaching of mathematics. He showed that the Delian Problem could be reduced to finding two mean proportionals between a given line and another twice as long. For, in the proportion $a : x = x : y = y : 2a$, since $x^2 = ay$ and $y^2 = 2ax$ and $x^4 = a^2y^2$, we have $x^4 = 2a^3x$ and $x^3 = 2a^3$. But,

of course, he failed to find the two mean proportionals by geometric construction with ruler and compasses. He made himself celebrated by squaring certain lunes. According to Simplicius, Hippocrates believed he actually succeeded in applying one of his lune-quadratures to the quadrature of the circle. That Hippocrates really committed this fallacy is not generally accepted.

In the first lune which he squared, he took an isosceles triangle ABC, right-angled at C, and drew a semi-circle on AB as a diameter, and passing through C. He drew also a semi-circle on AC as a diameter and lying outside the triangle ABC. The lunar area thus formed is half the area of the triangle ABC. This is the first example of a curvilinear area which admits of exact quadrature. Hippocrates squared other lunes, hoping, no doubt, that he might be led to the quadrature of the circle.[1] In 1840 Th. Clausen found other quadrable lunes, but in 1902 E. Landau of Göttingen pointed out that two of the four lunes which Clausen supposed to be new, were known to Hippocrates.[2]

In his study of the quadrature and duplication-problems, Hippocrates contributed much to the geometry of the circle. He showed that circles are to each other as the squares of their diameters, that similar segments in a circle are as the squares of their chords and contain equal angles, that in a segment less than a semi-circle the angle is obtuse. Hippocrates contributed vastly to the logic of geometry. His investigations are the oldest "reasoned geometrical proofs in existence" (Gow). For the purpose of describing geometrical figures he used letters, a practice probably introduced by the Pythagoreans.

The subject of similar figures, as developed by Hippocrates, involved the theory of proportion. Proportion had, thus far, been used by the Greeks only in numbers. They never succeeded in uniting the notions of numbers and magnitudes. The term "number" was used by them in a restricted sense. What we call irrational numbers was not included under this notion. Not even rational fractions were called numbers. They used the word in the same sense as we use "positive integers." Hence numbers were conceived as *discontinuous*, while magnitudes were *continuous*. The two notions appeared, therefore, entirely distinct. The chasm between them is exposed to full view in the statement of Euclid that "incommensurable magnitudes do not have the same ratio as numbers." In Euclid's *Elements* we find the theory of proportion of magnitudes developed and treated independent of that of numbers. The transfer of the theory of proportion from numbers to magnitudes (and to lengths in particular) was a difficult and important step.

[1] A full account is given by G. Loria in his *Le scienze esatte nell'antica Grecia*, Milano, 2 edition, 1914, pp. 74–94. Loria gives also full bibliographical references to the extensive literature on Hippocrates.

[2] E. W. Hobson, *Squaring the Circle*, Cambridge, 1913, p. 16.

Hippocrates added to his fame by writing a geometrical text-book, called the *Elements*. This publication shows that the Pythagorean habit of secrecy was being abandoned; secrecy was contrary to the spirit of Athenian life.

The sophist **Antiphon,** a contemporary of Hippocrates, introduced the *process* of exhaustion for the purpose of solving the problem of the quadrature. He did himself credit by remarking that by inscribing in a circle a square or an equilateral triangle, and on its sides erecting isosceles triangles with their vertices in the circumference, and on the sides of these triangles erecting new triangles, etc., one could obtain a succession of regular polygons, of which each approaches nearer to the area of the circle than the previous one, until the circle is finally *exhausted*. Thus is obtained an inscribed polygon whose sides coincide with the circumference. Since there can be found squares equal in area to any polygon, there also can be found a square equal to the last polygon inscribed, and therefore equal to the circle itself. **Bryson of Heraclea,** a contemporary of Antiphon, advanced the problem of the quadrature considerably by circumscribing polygons at the same time that he inscribed polygons. He erred, however, in assuming that the area of a circle was the arithmetical mean between circumscribed and inscribed polygons. Unlike Bryson and the rest of Greek geometers, Antiphon seems to have believed it possible, by continually doubling the sides of an inscribed polygon, to obtain a polygon coinciding with the circle. This question gave rise to lively disputes in Athens. If a polygon can coincide with the circle, then, says Simplicius, we must put aside the notion that magnitudes are divisible *ad infinitum*. This difficult philosophical question led to paradoxes that are difficult to explain and that deterred Greek mathematicians from introducing ideas of infinity into their geometry; rigor in geometric proofs demanded the exclusion of obscure conceptions. Famous are the arguments against the possibility of motion that were advanced by **Zeno** of Elea, the great dialectician (early in the 5th century B. C.). None of Zeno's writings have come down to us. We know of his tenets only through his critics, Plato, Aristotle, Simplicius. Aristotle, in his *Physics*, VI, 9, ascribes to Zeno four arguments, called "Zeno's paradoxies": (1) The "Dichotomy": You cannot traverse an infinite number of points in a finite time; you must traverse the half of a given distance before you traverse the whole, and the half of that again before you can traverse the whole. This goes on *ad infinitum*, so that (if space is made up of points) there is an infinite number in any given space, and it cannot be traversed in a finite time. (2) The "Achilles": Achilles cannot overtake a tortoise. For, Achilles must first reach the place from which the tortoise started. By that time the tortoise will have moved on a little way. Achilles must then traverse that, and still the tortoise will be ahead. He is always nearer, yet never makes up to it. (3) The "Arrow":

An arrow in any given moment of its flight must be at rest in some particular point. (4) The "Stade": Suppose three parallel rows of points in juxtaposition, as in Fig. 1. One of these (B) is immovable,

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A . . . .          ←A . . . .
B . . . .           B   . . . .
C . . . .           C       . . . .→
  Fig. 1                Fig. 2
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while A and C move in opposite directions with equal velocity, so as to come into the position in Fig. 2. The movement of C relatively to A will be double its movement relatively to B, or, in other words, any given point in C has passed twice as many points in A as it has in B. It cannot, therefore, be the case that an instant of time corresponds to the passage from one point to another.

Plato says that Zeno's purpose was "to protect the arguments of Parmenides against those who make fun of him"; Zeno argues that "there is no *many*," he "denies plurality." That Zeno's reasoning was wrong has been the view universally held since the time of Aristotle down to the middle of the nineteenth century. More recently the opinion has been advanced that Zeno was incompletely and incorrectly reported, that his arguments are serious efforts, conducted with logical rigor. This view has been advanced by Cousin, Grote and P. Tannery.[1] Tannery claims that Zeno did not deny motion, but wanted to show that motion was impossible under the Pythagorean conception of space as the sum of points, that the four arguments must be taken together as constituting a dialogue between Zeno and an adversary and that the arguments are in the form of a double dilemma into which Zeno forces his adversary. Zeno's arguments involve concepts of continuity, of the infinite and infinitesimal; they are as much the subjects of debate now as they were in the time of Aristotle. Aristotle did not successfully explain Zeno's paradoxes. He gave no reply to the query arising in the mind of the student, how is it possible for a variable to reach its limit? Aristotle's continuum was a sensuous, physical one; he held that, since a line cannot be built up of points, a line cannot actually be subdivided into points. "The continued bisection of a quantity is unlimited, so that the unlimited exists potentially, but is actually never reached." No satisfactory explanation of Zeno's arguments was given before the creation of Georg Cantor's continuum and theory of aggregates.

The process of exhaustion due to Antiphon and Bryson gave rise to the cumbrous but perfectly rigorous "method of exhaustion." In determining the ratio of the areas between two curvilinear plane figures, say two circles, geometers first inscribed or circumscribed similar polygons, and then by increasing indefinitely the number of

[1] See F. Cajori, "The History of Zeno's Arguments on Motion" in the *Americ. Math. Monthly*, Vol. 22, 1915, p. 3.

sides, nearly exhausted the spaces between the polygons and circumferences. From the theorem that similar polygons inscribed in circles are to each other as the squares on their diameters, geometers may have divined the theorem attributed to Hippocrates of Chios that the circles, which differ but little from the last drawn polygons, must be to each other as the squares on their diameters. But in order to exclude all vagueness and possibility of doubt, later Greek geometers applied reasoning like that in Euclid, XII, 2, as follows: Let C and c, D and d be respectively the circles and diameters in question. Then if the proportion $D^2 : d^2 = C : c$ is not true, suppose that $D^2 : d^2 = C : c^1$. If $c^1 < c$, then a polygon p can be inscribed in the circle c which comes nearer to it in area than does c^1. If P be the corresponding polygon in C, then $P : p = D^2 : d^2 = C : c^1$, and $P : C = p : c^1$. Since $p > c^1$, we have $P > C$, which is absurd. Next they proved by this same method of *reductio ad absurdum* the falsity of the supposition that $c^1 > c$. Since c^1 can be neither larger nor smaller than c, it must be equal to it, Q.E.D. Hankel refers this Method of Exhaustion back to Hippocrates of Chios, but the reasons for assigning it to this early writer, rather than to Eudoxus, seem insufficient.

Though progress in geometry at this period is traceable only at Athens, yet Ionia, Sicily, Abdera in Thrace, and Cyrene produced mathematicians who made creditable contributions to the science. We can mention here only **Democritus of Abdera** (about 460–370 B. C.), a pupil of Anaxagoras, a friend of Philolaus, and an admirer of the Pythagoreans. He visited Egypt and perhaps even Persia. He was a successful geometer and wrote on incommensurable lines, on geometry, on numbers, and on perspective. None of these works are extant. He used to boast that in the construction of plane figures with proof no one had yet surpassed him, not even the so-called harpedonaptæ ("rope-stretchers") of Egypt. By this assertion he pays a flattering compliment to the skill and ability of the Egyptians.

The Platonic School

During the Peloponnesian War (431–404 B. C.) the progress of geometry was checked. After the war, Athens sank into the background as a minor political power, but advanced more and more to the front as the leader in philosophy, literature, and science. **Plato** was born at Athens in 429 B. C., the year of the great plague, and died in 348. He was a pupil and near friend of Socrates, but it was not from him that he acquired his taste for mathematics. After the death of Socrates, Plato travelled extensively. In Cyrene he studied mathematics under Theodorus. He went to Egypt, then to Lower Italy and Sicily, where he came in contact with the Pythagoreans. Archytas of Tarentum and Timæus of Locri became his intimate friends. On his return to Athens, about 389 B. C., he founded his school in the

groves of the *Academia*, and devoted the remainder of his life to teaching and writing.

Plato's physical philosophy is partly based on that of the Pythagoreans. Like them, he sought in arithmetic and geometry the key to the universe. When questioned about the occupation of the Deity, Plato answered that "He geometrises continually." Accordingly, a knowledge of geometry is a necessary preparation for the study of philosophy. To show how great a value he put on mathematics and how necessary it is for higher speculation, Plato placed the inscription over his porch, "Let no one who is unacquainted with geometry enter here." Xenocrates, a successor of Plato as teacher in the Academy, followed in his master's footsteps, by declining to admit a pupil who had no mathematical training, with the remark, "Depart, for thou hast not the grip of philosophy." Plato observed that geometry trained the mind for correct and vigorous thinking. Hence it was that the *Eudemian Summary* says, "He filled his writings with mathematical discoveries, and exhibited on every occasion the remarkable connection between mathematics and philosophy."

With Plato as the head-master, we need not wonder that the Platonic school produced so large a number of mathematicians. Plato did little real original work, but he made valuable improvements in the logic and methods employed in geometry. It is true that the Sophist geometers of the previous century were fairly rigorous in their proofs, but as a rule they did not reflect on the inward nature of their methods. They used the axioms without giving them explicit expression, and the geometrical concepts, such as the point, line, surface, etc., without assigning to them formal definitions.[1] The Pythagoreans called a point "unity in position," but this is a statement of a philosophical theory rather than a definition. Plato objected to calling a point a "geometrical fiction." He defined a point as the "beginning of a line" or as "an indivisible line," and a line as "length without breadth." He called the point, line, surface, the "boundaries" of the line, surface, solid, respectively. Many of the definitions in Euclid are to be ascribed to the Platonic school. The same is probably true of Euclid's axioms. Aristotle refers to Plato the axiom that "equals subtracted from equals leave equals."

One of the greatest achievements of Plato and his school is the invention of *analysis* as a method of proof. To be sure, this method had been used unconsciously by Hippocrates and others; but Plato, like a true philosopher, turned the instinctive logic into a conscious, legitimate method.

[1] "If any one scientific invention can claim pre-eminence over all others, I should be inclined myself to erect a monument to the unknown inventor of the mathematical point, as the supreme type of that process of abstraction which has been a necessary condition of scientific work from the very beginning." *Horace Lamb's* Address, Section A, Brit. Ass'n, 1904.

The terms *synthesis* and *analysis* are used in mathematics in a more special sense than in logic. In ancient mathematics they had a different meaning from what they now have. The oldest definition of mathematical analysis as opposed to synthesis is that given in Euclid, XIII, 5, which in all probability was framed by Eudoxus: "Analysis is the obtaining of the thing sought by assuming it and so reasoning up to an admitted truth; synthesis is the obtaining of the thing sought by reasoning up to the inference and proof of it." The analytic method is not conclusive, unless all operations involved in it are known to be reversible. To remove all doubt, the Greeks, as a rule, added to the analytic process a synthetic one, consisting of a reversion of all operations occurring in the analysis. Thus the aim of analysis was to aid in the discovery of synthetic proofs or solutions.

Plato is said to have solved the problem of the duplication of the cube. But the solution is open to the very same objection which he made to the solutions by Archytas, Eudoxus, and Menæchmus. He called their solutions not geometrical, but mechanical, for they required the use of other instruments than the ruler and compasses. He said that thereby "the good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God." These objections indicate either that the solution is wrongly attributed to Plato or that he wished to show how easily non-geometric solutions of that character can be found. It is now rigorously established that the duplication problem, as well as the trisection and quadrature problems, cannot be solved by means of the ruler and compasses only.

Plato gave a healthful stimulus to the study of stereometry, which until his time had been entirely neglected by the Greeks. The sphere and the regular solids have been studied to some extent, but the prism, pyramid, cylinder, and cone were hardly known to exist. All these solids became the subjects of investigation by the Platonic school. One result of these inquiries was epoch-making. **Menæchmus,** an associate of Plato and pupil of Eudoxus, invented the conic sections, which, in course of only a century, raised geometry to the loftiest height which it was destined to reach during antiquity. Menæchmus cut three kinds of cones, the "right-angled," "acute-angled," and "obtuse-angled," by planes at right angles to a side of the cones, and thus obtained the three sections which we now call the parabola, ellipse, and hyperbola. Judging from the two very elegant solutions of the "Delian Problem" by means of intersections of these curves, Menæchmus must have succeeded well in investigating their properties. In what manner he carried out the graphic construction of these curves is not known.

Another great geometer was **Dinostratus,** the brother of Menæch-

mus and pupil of Plato. Celebrated is his mechanical solution of the quadrature of the circle, by means of the *quadratrix* of Hippias.

Perhaps the most brilliant mathematician of this period was **Eudoxus.** He was born at Cnidus about 408 B. C., studied under Archytas, and later, for two months, under Plato. He was imbued with a true spirit of scientific inquiry, and has been called the father of scientific astronomical observation. From the fragmentary notices of his astronomical researches, found in later writers, Ideler and Schiaparelli succeeded in reconstructing the system of Eudoxus with its celebrated representation of planetary motions by "concentric spheres." Eudoxus had a school at Cyzicus, went with his pupils to Athens, visiting Plato, and then returned to Cyzicus, where he died 355 B. C. The fame of the academy of Plato is to a large extent due to Eudoxus's pupils of the school at Cyzicus, among whom are Menæchmus, Dinostratus, Athenæus, and Helicon. Diogenes Laertius describes Eudoxus as astronomer, physician, legislator, as well as geometer. The *Eudemian Summary* says that Eudoxus "first increased the number of general theorems, added to the three proportions three more, and raised to a considerable quantity the learning, begun by Plato, on the subject of the section, to which he applied the analytical method." By this "section" is meant, no doubt, the "golden section" (*sectio aurea*), which cuts a line in extreme and mean ratio. The first five propositions in Euclid XIII relate to lines cut by this section, and are generally attributed to Eudoxus. Eudoxus added much to the knowledge of solid geometry. He proved, says Archimedes, that a pyramid is exactly one-third of a prism, and a cone one-third of a cylinder, having equal base and altitude. The proof that spheres are to each other as the cubes of their radii is probably due to him. He made frequent and skilful use of the method of exhaustion, of which he was in all probability the inventor. A scholiast on Euclid, thought to be Proclus, says further that Eudoxus practically invented the whole of Euclid's fifth book. Eudoxus also found two mean proportionals between two given lines, but the method of solution is not known.

Plato has been called a maker of mathematicians. Besides the pupils already named, the *Eudemian Summary* mentions the following: **Theætetus** of Athens, a man of great natural gifts, to whom, no doubt, Euclid was greatly indebted in the composition of the 10th book,[1] treating of incommensurables and of the 13th book; **Leodamas** of Thasos; **Neocleides** and his pupil **Leon,** who added much to the work of their predecessors, for Leon wrote an *Elements* carefully designed, both in number and utility of its proofs; **Theudius of Magnesia,** who composed a very good book of *Elements* and generalised propositions, which had been confined to particular cases; **Hermotimus of Colophon,** who discovered many propositions of the *Elements* and com-

[1] G. J. Allman, *op. cit.*, p. 212.

posed some on *loci;* and, finally, the names of **Amyclas of Heraclea, Cyzicenus of Athens,** and **Philippus of Mende.**

A skilful mathematician of whose life and works we have no details is **Aristæus,** the elder, probably a senior contemporary of Euclid. The fact that he wrote a work on conic sections tends to show that much progress had been made in their study during the time of Menæchmus. Aristæus wrote also on regular solids and cultivated the analytic method. His works contained probably a summary of the researches of the Platonic school.[1]

Aristotle (384–322 B. C.), the systematiser of deductive logic, though not a professed mathematician, promoted the science of geometry by improving some of the most difficult definitions. His *Physics* contains passages with suggestive hints of the principle of virtual velocities. He gave the best discussion of continuity and of Zeno's arguments against motion, found in antiquity. About his time there appeared a work called *Mechanica,* of which he is regarded by some as the author. Mechanics was totally neglected by the Platonic school.

The First Alexandrian School

In the previous pages we have seen the birth of geometry in Egypt, its transference to the Ionian Islands, thence to Lower Italy and to Athens. We have witnessed its growth in Greece from feeble childhood to vigorous manhood, and now we shall see it return to the land of its birth and there derive new vigor.

During her declining years, immediately following the Peloponnesian War, Athens produced the greatest scientists and philosophers of antiquity. It was the time of Plato and Aristotle. In 338 B. C., at the battle of Chæronea, Athens was beaten by Philip of Macedon, and her power was broken forever. Soon after, Alexander the Great, the son of Philip, started out to conquer the world. In eleven years he built up a great empire which broke to pieces in a day. Egypt fell to the lot of Ptolemy Soter. Alexander had founded the seaport of Alexandria, which soon became the "noblest of all cities." Ptolemy made Alexandria the capital. The history of Egypt during the next three centuries is mainly the history of Alexandria. Literature, philosophy, and art were diligently cultivated. Ptolemy created the university of Alexandria. He founded the great Library and built laboratories, museums, a zoölogical garden, and promenades. Alexandria soon became the great centre of learning.

Demetrius Phalereus was invited from Athens to take charge of the Library, and it is probable, says Gow, that **Euclid** was invited with him to open the mathematical school. According to the studies of H. Vogt,[2] Euclid was born about 365 B. C. and wrote his *Elements*

[1] G. J. Allman, *op. cit.*, p. 205.

[2] *Bibliotheca mathematica*, 3 S., Vol. 13, 1913, pp. 193–202.

between 330 and 320 B. C. Of the life of Euclid, little is known, except what is added by Proclus to the *Eudemian Summary*. Euclid, says Proclus, was younger than Plato and older than Eratosthenes and Archimedes, the latter of whom mentions him. He was of the Platonic sect, and well read in its doctrines. He collected the *Elements*, put in order much that Eudoxus had prepared, completed many things of Theætetus, and was the first who reduced to unobjectionable demonstration the imperfect attempts of his predecessors. When Ptolemy once asked him if geometry could not be mastered by an easier process than by studying the *Elements*, Euclid returned the answer, "There is no royal road to geometry." Pappus states that Euclid was distinguished by the fairness and kindness of his disposition, particularly toward those who could do anything to advance the mathematical sciences. Pappus is evidently making a contrast to Apollonius, of whom he more than insinuates the opposite character.[1] A pretty little story is related by Stobæus:[2] "A youth who had begun to read geometry with Euclid, when he had learnt the first proposition, inquired, 'What do I get by learning these things?' So Euclid called his slave and said, 'Give him threepence, since he must make gain out of what he learns.'" These are about all the personal details preserved by Greek writers. Syrian and Arabian writers claim to know much more, but they are unreliable. At one time Euclid of Alexandria was universally confounded with Euclid of Megara, who lived a century earlier.

The fame of Euclid has at all times rested mainly upon his book on geometry, called the *Elements*. This book was so far superior to the *Elements* written by Hippocrates, Leon, and Theudius, that the latter works soon perished in the struggle for existence. The Greeks gave Euclid the special title of "the author of the *Elements*." It is a remarkable fact in the history of geometry, that the *Elements* of Euclid, written over two thousand years ago, are still regarded by some as the best introduction to the mathematical sciences. In England they were used until the present century extensively as a text-book in schools. Some editors of Euclid have, however, been inclined to credit him with more than is his due. They would have us believe that a finished and unassailable system of geometry sprang at once from the brain of Euclid, "an armed Minerva from the head of Jupiter." They fail to mention the earlier eminent mathematicians from whom Euclid got his material. Comparatively few of the propositions and proofs in the *Elements* are his own discoveries. In fact, the proof of the "Theorem of Pythagoras" is the only one directly ascribed to him. Allman conjectures that the substance of Books I, II, IV comes from the Pythagoreans, that the substance of Book VI is due to the Pytha-

[1] A. De Morgan, "Eucleides" in *Smith's Dictionary of Greek and Roman Biography and Mythology*.

[2] J. Gow, *op. cit.*, p. 195.

goreans and Eudoxus, the latter contributing the doctrine of proportion as applicable to incommensurables and also the Method of Exhaustions (Book XII), that Theætetus contributed much toward Books X and XIII, that the principal part of the original work of Euclid himself is to be found in Book X.[1] Euclid was the greatest systematiser of his time. By careful selection from the material before him, and by logical arrangement of the propositions selected, he built up, from a few definitions and axioms, a proud and lofty structure. It would be erroneous to believe that he incorporated into his *Elements* all the elementary theorems known at his time. Archimedes, Apollonius, and even he himself refer to theorems not included in his *Elements*, as being well-known truths.

The text of the *Elements* that was commonly used in schools was Theon's edition. Theon of Alexandria, the father of Hypatia, brought out an edition, about 700 years after Euclid, with some alterations in the text. As a consequence, later commentators, especially Robert Simson, who labored under the idea that Euclid must be absolutely perfect, made Theon the scapegoat for all the defects which they thought they could discover in the text as they knew it. But among the manuscripts sent by Napoleon I from the Vatican to Paris was found a copy of the *Elements* believed to be anterior to Theon's recension. Many variations from Theon's version were noticed therein, but they were not at all important, and showed that Theon generally made only verbal changes. The defects in the *Elements* for which Theon was blamed must, therefore, be due to Euclid himself. The *Elements* used to be considered as offering models of scrupulously rigorous demonstrations. It is certainly true that in point of rigor it compares favorably with its modern rivals; but when examined in the light of strict mathematical logic, it has been pronounced by C. S. Peirce to be "riddled with fallacies." The results are correct only because the writer's experience keeps him on his guard. In many proofs Euclid relies partly upon intuition.

At the beginning of our editions of the *Elements*, under the head of definitions, are given the assumptions of such notions as the point, line, etc., and some verbal explanations. Then follow three postulates or demands, and twelve axioms. The term "axiom" was used by Proclus, but not by Euclid. He speaks, instead, of "common notions"—common either to all men or to all sciences. There has been much controversy among ancient and modern critics on the postulates and axioms. An immense preponderance of manuscripts and the testimony of Proclus place the "axioms" about *right angles* and *parallels* among the postulates.[2] This is indeed their proper place,

[1] G. J. Allman, *op. cit.*, p. 211.

[2] A. De Morgan, *loc. cit.;* H. Hankel, *Theorie der Complexen Zahlensysteme*, Leipzig, 1867, p. 52. In the various editions of Euclid's *Elements* different numbers are assigned to the axioms. Thus the parallel axiom is called by Robert Simson the

for they are really *assumptions*, and not *common notions* or axioms. The postulate about *parallels* plays an important rôle in the history of non-Euclidean geometry. An important postulate which Euclid missed was the one of superposition, according to which figures can be moved about in space without any alteration in form or magnitude.

The *Elements* contains thirteen books by Euclid, and two, of which it is supposed that Hypsicles and Damascius are the authors. The first four books are on plane geometry. The fifth book treats of the theory of proportion as applied to magnitudes in general. It has been greatly admired because of its rigor of treatment. Beginners find the book difficult. Expressed in modern symbols, Euclid's definition of proportion is thus: Four magnitudes, a, b, c, d, are in proportion, when for any integers m and n, we have simultaneously $ma \gtreqless nb$, and $mc \gtreqless nd$. Says T. L. Heath,[1] "certain it is that there is an exact correspondence, almost coincidence, between Euclid's definition of equal ratios and the modern theory of irrationals due to Dedekind. H. G. Zeuthen finds a close resemblance between Euclid's definition and Weierstrass' definition of equal numbers. The sixth book develops the geometry of similar figures. Its 27th Proposition is the earliest maximum theorem known to history. The seventh, eighth, ninth books are on the theory of numbers, or on arithmetic. According to P. Tannery, the knowledge of the existence of irrationals must have greatly affected the mode of writing the *Elements*. The old naïve theory of proportion being recognized as untenable, proportions are not used at all in the first four books. The rigorous theory of Eudoxus was postponed as long as possible, because of its difficulty. The interpolation of the arithmetical books VII–IX is explained as a preparation for the fuller treatment of the irrational in book X. Book VII explains the G. C. D. of two numbers by the process of division (the so-called "Euclidean method"). The theory of proportion of (rational) numbers is then developed on the basis of the definition, "Numbers are proportional when the first is the same multiple, part, or parts of the second that the third is of the fourth." This is believed to be the older, Pythagorean theory of proportion.[2] The tenth treats of the theory of incommensurables. De Morgan considered this the most wonderful of all. We give a fuller account of it under the head of Greek Arithmetic. The next three books are on

12th, by Bolyai the 11th, by Clavius the 13th, by F. Peyrard the 5th. It is called the 5th *postulate* in old manuscripts, also by Heiberg and Menge in their annotated edition of Euclid's works, in Greek and Latin, Leipzig, 1883, and by T. L. Heath in his *Thirteen Books of Euclid's Elements*, Vols. I–III, Cambridge, 1908. Heath's is the most recent translation into English and is very fully and ably annotated.

[1] T. L. Heath, *op. cit.*, Vol. II, p. 124.

[2] Read H. B. Fine, "Ratio, Proportion and Measurement in the Elements of Euclid," *Annals of Mathematics*, Vol. XIX, 1917, pp. 70–76.

stereometry. The eleventh contains its more elementary theorems; the twelfth, the metrical relations of the pyramid, prism, cone, cylinder, and sphere. The thirteenth treats of the regular polygons, especially of the triangle and pentagon, and then uses them as faces of the five regular solids; namely, the tetraedron, octaedron, icosaedron, cube, and dodecaedron. The regular solids were studied so extensively by the Platonists that they received the name of "Platonic figures." The statement of Proclus that the whole aim of Euclid in writing the *Elements* was to arrive at the construction of the regular solids, is obviously wrong. The fourteenth and fifteenth books, treating of solid geometry, are apocryphal. It is interesting to see that to Euclid, and to Greek mathematicians in general, the existence of areas was evident from intuition. The notion of non-quadrable areas had not occurred to them.

A remarkable feature of Euclid's, and of all Greek geometry before Archimedes is that it eschews mensuration. Thus the theorem that the area of a triangle equals half the product of its base and its altitude is foreign to Euclid.

Another extant book of Euclid is the *Data*. It seems to have been written for those who, having completed the *Elements*, wish to acquire the power of solving new problems proposed to them. The *Data* is a course of practice in *analysis*. It contains little or nothing that an intelligent student could not pick up from the *Elements* itself. Hence it contributes little to the stock of scientific knowledge. The following are the other works with texts more or less complete and generally attributed to Euclid: *Phænomena*, a work on spherical geometry and astronomy; *Optics*, which develops the hypothesis that light proceeds from the eye, and not from the object seen; *Catoptrica*, containing propositions on reflections from mirrors: *De Divisionibus*, a treatise on the division of plane figures into parts having to one another a given ratio; [1] *Sectio Canonis*, a work on musical intervals. His treatise on *Porisms* is lost; but much learning has been expended by Robert Simson and M. Chasles in restoring it from numerous notes found in the writings of Pappus. The term "porism" is vague in meaning. According to Proclus, the aim of a porism is not to state some property or truth, like a theorem, nor to effect a construction, like a problem, but to find and bring to view a thing which necessarily exists with given numbers or a given construction, as, to find the centre of a given circle, or to find the G. C. D. of two given numbers. Porisms, according to Chasles, are incomplete theorems, "expressing certain relations between things variable according to a common law." Euclid's other lost works are *Fallacies*, containing exercises in detection of fallacies; *Conic Sections*, in four books, which are the foundation of a work on the same subject by Apollonius; and *Loci on a Surface*,

[1] A careful restoration was brought out in 1915 by R. C. Archibald of Brown University.

the meaning of which title is not understood. Heiberg believes it to mean "loci which are surfaces."

The immediate successors of Euclid in the mathematical school at Alexandria were probably **Conon, Dositheus,** and **Zeuxippus,** but little is known of them.

Archimedes (287?–212 B. C.), the greatest mathematician of antiquity, was born in Syracuse. Plutarch calls him a relation of King Hieron; but more reliable is the statement of Cicero, who tells us he was of low birth. Diodorus says he visited Egypt, and, since he was a great friend of Conon and Eratosthenes, it is highly probable that he studied in Alexandria. This belief is strengthened by the fact that he had the most thorough acquaintance with all the work previously done in mathematics. He returned, however, to Syracuse, where he made himself useful to his admiring friend and patron, King Hieron, by applying his extraordinary inventive genius to the construction of various war-engines, by which he inflicted much loss on the Romans during the siege of Marcellus. The story that, by the use of mirrors reflecting the sun's rays, he set on fire the Roman ships, when they came within bow-shot of the walls, is probably a fiction. The city was taken at length by the Romans, and Archimedes perished in the indiscriminate slaughter which followed. According to tradition, he was, at the time, studying the diagram to some problem drawn in the sand. As a Roman soldier approached him, he called out, "Don't spoil my circles." The soldier, feeling insulted, rushed upon him and killed him. No blame attaches to the Roman general Marcellus, who admired his genius, and raised in his honor a tomb bearing the figure of a sphere inscribed in a cylinder. When Cicero was in Syracuse, he found the tomb buried under rubbish.

Archimedes was admired by his fellow-citizens chiefly for his mechanical inventions; he himself prized far more highly his discoveries in pure science. He declared that "every kind of art which was connected with daily needs was ignoble and vulgar." Some of his works have been lost. The following are the extant books, arranged approximately in chronological order: 1. Two books on *Equiponderance of Planes* or *Centres of Plane Gravities*, between which is inserted his treatise on the *Quadrature of the Parabola;* 2. *The Method;* 3. Two books on the *Sphere* and *Cylinder;* 4. The *Measurement of the Circle;* 5. *On Spirals;* 6. *Conoids* and *Spheroids;* 7. The *Sand-Counter;* 8. Two books on *Floating Bodies;* 9. Fifteen *Lemmas*.

In the book on the *Measurement of the Circle*, Archimedes proves first that the area of a circle is equal to that of a right triangle having the length of the circumference for its base, and the radius for its altitude. In this he assumes that there exists a straight line equal in length to the circumference—an assumption objected to by some ancient critics, on the ground that it is not evident that a straight line can equal a curved one. The finding of such a line was the next

problem. He first finds an upper limit to the ratio of the circumference to the diameter, or π. To do this, he starts with an equilateral triangle of which the base is a tangent and the vertex is the centre of the circle. By successively bisecting the angle at the centre, by comparing ratios, and by taking the irrational square roots always a little too small, he finally arrived at the conclusion that $\pi < 3\frac{1}{7}$. Next he finds a lower limit by inscribing in the circle regular polygons of 6, 12, 24, 48, 96 sides, finding for each successive polygon its perimeter, which is, of course, always less than the circumference. Thus he finally concludes that "the circumference of a circle exceeds three times its diameter by a part which is less than $\frac{1}{7}$ but more than $\frac{10}{71}$ of the diameter." This approximation is exact enough for most purposes.

The *Quadrature of the Parabola* contains two solutions to the problem—one mechanical, the other geometrical. The method of exhaustion is used in both.

It is noteworthy that, perhaps through the influence of Zeno, infinitesimals (infinitely small constants) were not used in rigorous demonstration. In fact, the great geometers of the period now under consideration resorted to the radical measure of excluding them from demonstrative geometry by a postulate. This was done by Eudoxus, Euclid, and Archimedes. In the preface to the *Quadrature of the Parabola*, occurs the so-called "Archimedean postulate," which Archimedes himself attributes to Eudoxus: "When two spaces are unequal, it is possible to add to itself the difference by which the lesser is surpassed by the greater, so often that every finite space will be exceeded." Euclid (*Elements* V, 4) gives the postulate in the form of a definition: "Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other." Nevertheless, infinitesimals may have been used in tentative researches. That such was the case with Archimedes is evident from his book, *The Method*, formerly thought to be irretrievably lost, but fortunately discovered by Heiberg in 1906 in Constantinople. The contents of this book shows that he considered infinitesimals sufficiently scientific to suggest the truths of theorems, but not to furnish rigorous proofs. In finding the areas of parabolic segments, the volumes of spherical segments and other solids of revolution, he uses a mechanical process, consisting of the weighing of infinitesimal elements, which he calls straight lines or plane areas, but which are really infinitely narrow strips or infinitely thin plane laminæ.[1] The breadth or thickness is regarded as being the same in the elements weighed at any one time. The Archimedean postulate did not command the interest of mathematicians until the modern arithmetic continuum was created. It was O. Stolz that showed that it was a consequence of Dedekind's postulate relating to "sections."

[1] T. L. Heath, *Method of Archimedes*, Cambridge, 1912, p. 8.

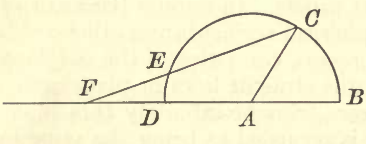
It would seem that, in his great researches, Archimedes' mode of procedure was, to start with mechanics (centre of mass of surfaces and solids) and by his infinitesimal-mechanical method to discover new results for which later he deduced and published the rigorous proofs. Archimedes knew the integral[1] $\int x^3 dx$.

Archimedes studied also the ellipse and accomplished its quadrature, but to the hyperbola he seems to have paid less attention. It is believed that he wrote a book on conic sections.

Of all his discoveries Archimedes prized most highly those in his *Sphere and Cylinder*. In it are proved the new theorems, that the surface of a sphere is equal to four times a great circle; that the surface segment of a sphere is equal to a circle whose radius is the straight line drawn from the vertex of the segment to the circumference of its basal circle; that the volume and the surface of a sphere are $\frac{2}{3}$ of the volume and surface, respectively, of the cylinder circumscribed about the sphere. Archimedes desired that the figure to the last proposition be inscribed on his tomb. This was ordered done by Marcellus.

The spiral now called the "spiral of Archimedes," and described in the book *On Spirals*, was discovered by Archimedes, and not, as some believe, by his friend Conon.[2] His treatise thereon is, perhaps, the most wonderful of all his works. Nowadays, subjects of this kind are made easy by the use of the infinitesimal calculus. In its stead the ancients used the method of exhaustion. Nowhere is the fertility of his genius more grandly displayed than in his masterly use of this method. With Euclid and his predecessors the method of exhaustion was only the means of proving propositions which must have been seen and believed before they were proved. But in the hands of Archimedes this method, perhaps combined with his infinitesimal-mechanical method, became an instrument of discovery.

By the word "conoid," in his book on *Conoids and Spheroids*, is meant the solid produced by the revolution of a parabola or a hyperbola about its axis. Spheroids are produced by the revolution of an ellipse, and are long or flat, according as the ellipse revolves around the major or minor axis. The book leads up to the cubature of these solids. A few constructions of geometric figures were given by Archimedes and Appolonius which were effected by "insertions." In the following trisection of an angle, attributed by the Arabs to Archimedes, the "insertion" is achieved by the aid of a *graduated* ruler.[3] To trisect the angle CAB, draw the arc BCD. Then "insert" the



[1] H. G. Zeuthen in *Bibliotheca mathematica*, 3 S., Vol. 7, 1906–7, p. 347.
[2] M. Cantor, *op. cit.*, Vol. I, 3 Aufl., 1907, p. 306.
[3] F. Enriques, *Fragen der Elementargeometrie*, deutsche Ausg. v. H. Fleischer, II, Leipzig, 1907, p. 234.

distance FE, equal to AB, marked on an edge passing through C and moved until the points E and F are located as shown in the figure. The required angle is EFD.

His arithmetical treatise and problems will be considered later. We shall now notice his works on mechanics. Archimedes is the author of the first sound knowledge on this subject. Archytas, Aristotle, and others attempted to form the known mechanical truths into a science, but failed. Aristotle knew the property of the lever, but could not establish its true mathematical theory. The radical and fatal defect in the speculations of the Greeks, in the opinion of Whewell, was "that though they had in their possession facts and ideas, *the ideas were not distinct and appropriate to the facts.*" For instance, Aristotle asserted that when a body at the end of a lever is moving, it may be considered as having two motions; one in the direction of the tangent and one in the direction of the radius; the former motion is, he says, *according to nature*, the latter *contrary to nature*. These inappropriate notions of "natural" and "unnatural" motions, together with the habits of thought which dictated these speculations, made the perception of the true grounds of mechanical properties impossible.[1] It seems strange that even after Archimedes had entered upon the right path, this science should have remained absolutely stationary till the time of Galileo—a period of nearly two thousand years.

The proof of the property of the lever, given in his *Equiponderance of Planes*, holds its place in many text-books to this day. Mach[2] criticizes it. "From the mere assumption of the equilibrium of equal weights at equal distances is derived the inverse proportionality of weight and lever arm! How is that possible?" Archimedes' estimate of the efficiency of the lever is expressed in the saying attributed to him, "Give me a fulcrum on which to rest, and I will move the earth."

While the *Equiponderance* treats of solids, or the equilibrium of solids, the book on *Floating Bodies* treats of hydrostatics. His attention was first drawn to the subject of specific gravity when King Hieron asked him to test whether a crown, professed by the maker to be pure gold, was not alloyed with silver. The story goes that our philosopher was in a bath when the true method of solution flashed on his mind. He immediately ran home, naked, shouting, "I have found it!" To solve the problem, he took a piece of gold and a piece of silver, each weighing the same as the crown. According to one author, he determined the volume of water displaced by the gold, silver, and crown respectively, and calculated from that the amount of gold and silver

[1] William Whewell, *History of the Inductive Sciences*, 3rd Ed., New York, 1858, Vol. I, p. 87. William Whewell (1794–1866) was Master of Trinity College, Cambridge.

[2] E. Mach, *The Science of Mechanics*, tr. by T. McCormack, Chicago, 1907, p. 14. Ernst Mach (1838–1916) was professor of the history and theory of the inductive sciences at the university of Vienna.

in the crown. According to another writer, he weighed separately the gold, silver, and crown, while immersed in water, thereby determining their loss of weight in water. From these data he easily found the solution. It is possible that Archimedes solved the problem by both methods.

After examining the writings of Archimedes, one can well understand how, in ancient times, an "Archimedean problem" came to mean a problem too deep for ordinary minds to solve, and how an "Archimedean proof" came to be the synonym for unquestionable certainty. Archimedes wrote on a very wide range of subjects, and displayed great profundity in each. He is the Newton of antiquity.

Eratosthenes, eleven years younger than Archimedes, was a native of Cyrene. He was educated in Alexandria under Callimachus the poet, whom he succeeded as custodian of the Alexandrian Library. His many-sided activity may be inferred from his works. He wrote on *Good and Evil*, *Measurement of the Earth*, *Comedy*, *Geography*, *Chronology*, *Constellations*, and the *Duplication of the Cube*. He was also a philologian and a poet. He measured the obliquity of the ecliptic and invented a device for finding prime numbers, to be described later. Of his geometrical writings we possess only a letter to Ptolemy Euergetes, giving a history of the duplication problem and also the description of a very ingenious mechanical contrivance of his own to solve it. In his old age he lost his eyesight, and on that account is said to have committed suicide by voluntary starvation.

About forty years after Archimedes flourished **Apollonius of Perga,** whose genius nearly equalled that of his great predecessor. He incontestably occupies the second place in distinction among ancient mathematicians. Apollonius was born in the reign of Ptolemy Euergetes and died under Ptolemy Philopator, who reigned 222–205 B. C. He studied at Alexandria under the successors of Euclid, and for some time, also, at Pergamum, where he made the acquaintance of that Eudemus to whom he dedicated the first three books of his *Conic Sections*. The brilliancy of his great work brought him the title of the "Great Geometer." This is all that is known of his life.

His *Conic Sections* were in eight books, of which the first four only have come down to us in the original Greek. The next three books were unknown in Europe till the middle of the seventeenth century, when an Arabic translation, made about 1250, was discovered. The eighth book has never been found. In 1710 E. Halley of Oxford published the Greek text of the first four books and a Latin translation of the remaining three, together with his conjectural restoration of the eighth book, founded on the introductory lemmas of Pappus. The first four books contain little more than the substance of what earlier geometers had done. Eutocius tells us that Heraclides, in his life of Archimedes, accused Appolonius of having appropriated, in his *Conic Sections*, the unpublished discoveries of that great mathematician.

It is difficult to believe that this charge rests upon good foundation. Eutocius quotes Geminus as replying that neither Archimedes nor Apollonius claimed to have invented the conic sections, but that Apollonius had introduced a real improvement. While the first three or four books were founded on the works of Menæchmus, Aristæus, Euclid, and Archimedes, the remaining ones consisted almost entirely of new matter. The first three books were sent to Eudemus at intervals, the other books (after Eudemus's death) to one Attalus. The preface of the second book is interesting as showing the mode in which Greek books were "published" at this time. It reads thus: "I have sent my son Apollonius to bring you (Eudemus) the second book of my Conics. Read it carefully and communicate it to such others as are worthy of it. If Philonides, the geometer, whom I introduced to you at Ephesus, comes into the neighbourhood of Pergamum, give it to him also." [1]

The first book, says Apollonius in his preface to it, "contains the mode of producing the three sections and the conjugate hyperbolas and their principal characteristics, more fully and generally worked out than in the writings of other authors." We remember that Menæchmus, and all his successors down to Apollonius, considered only sections of *right* cones by a plane perpendicular to their sides, and that the three sections were obtained each from a different cone. Apollonius introduced an important generalisation. He produced all the sections from one and the same cone, whether right or scalene, and by sections which may or may not be perpendicular to its sides. The old names for the three curves were now no longer applicable. Instead of calling the three curves, sections of the "acute-angled," "right-angled," and "obtuse-angled" cone, he called them *ellipse*, *parabola*, and *hyperbola*, respectively. To be sure, we find the words "parabola" and "ellipse" in the works of Archimedes, but they are probably only interpolations. The word "ellipse" was applied because $y^2<px$, p being the parameter; the word "parabola" was introduced because $y^2=px$, and the term "hyperbola" because $y^2>px$.

The treatise of Apollonius rests on a unique property of conic sections, which is derived directly from the nature of the cone in which these sections are found. How this property forms the key to the system of the ancients is told in a masterly way by M. Chasles.[2] "Conceive," says he, "an oblique cone on a circular base; the straight line drawn from its summit to the centre of the circle forming its base is called the *axis* of the cone. The plane passing through the axis, perpendicular to its base, cuts the cone along two lines and determines in the circle a diameter; the triangle having this diameter for its base

[1] H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Alterthum*, Kopenhagen, 1886, p. 502.

[2] M. Chasles, *Geschichte der Geometrie.* Aus dem Französischen übertragen durch, Dr. L. A. Sohncke, Halle, 1839, p. 15.

and the two lines for its sides, is called *the triangle through the axis.* In the formation of his conic sections, Apollonius supposed the cutting plane to be perpendicular to the plane of the triangle through the axis. The points in which this plane meets the two sides of this triangle are the *vertices* of the curve; and the straight line which joins these two points is a diameter of it. Apollonius called this diameter *latus transversum.* At one of the two vertices of the curve erect a perpendicular (*latus rectum*) to the plane of the triangle through the axis, of a certain length, to be determined as we shall specify later, and from the extremity of this perpendicular draw a straight line to the other vertex of the curve; now, through any point whatever of the diameter of the curve, draw at right angles an *ordinate:* the square of this ordinate, comprehended between the diameter and the curve, will be equal to the rectangle constructed on the portion of the ordinate comprised between the diameter and the straight line, and the part of the diameter comprised between the first vertex and the foot of the ordinate. Such is the characteristic property which Apollonius recognises in his conic sections and which he uses for the purpose of inferring from it, by adroit transformations and deductions, nearly all the rest. It plays, as we shall see, in his hands, almost the same rôle as the equation of the second degree with two variables (abscissa and ordinate) in the system of analytic geometry of Descartes." Apollonius made use of co-ordinates as did Menæchmus before him.[1] Chasles continues:

"It will be observed from this that the diameter of the curve and the perpendicular erected at one of its extremities suffice to construct the curve. These are the two elements which the ancients used, with which to establish their theory of conics. The perpendicular in question was called by them *latus erectum;* the moderns changed this name first to that of *latus rectum*, and afterwards to that of *parameter.*"

The first book of the *Conic Sections* of Apollonius is almost wholly devoted to the generation of the three principal conic sections.

The second book treats mainly of asymptotes, axes, and diameters.

The third book treats of the equality or proportionality of triangles, rectangles, or squares, of which the component parts are determined by portions of transversals, chords, asymptotes, or tangents, which are frequently subject to a great number of conditions. It also touches the subject of foci of the ellipse and hyperbola.

In the fourth book, Apollonius discusses the harmonic division of straight lines. He also examines a system of two conics, and shows that they cannot cut each other in more than four points. He investigates the various possible relative positions of two conics, as, for instance, when they have one or two points of contact with each other.

The fifth book reveals better than any other the giant intellect of its author. Difficult questions of *maxima and minima*, of which few

[1] T. L. Heath, *Apollonius of Perga*, Cambridge, 1896, p. CXV.

examples are found in earlier works, are here treated most exhaustively. The subject investigated is, to find the longest and shortest lines that can be drawn from a given point to a conic. Here are also found the germs of the subject of *evolutes* and *centres of osculation*.

The sixth book is on the similarity of conics.

The seventh book is on conjugate diameters.

The eighth book, as restored by Halley, continues the subject of conjugate diameters.

It is worthy of notice that Apollonius nowhere introduces the notion of *directrix* for a conic, and that, though he incidentally discovered the *focus* of an ellipse and hyperbola, he did not discover the focus of a parabola.[1] Conspicuous in his geometry is also the absence of technical terms and symbols, which renders the proofs long and cumbrous. R. C. Archibald claims that Apollonius was familiar with the centres of similitude of circles, usually attributed to Monge. T. L. Heath[2] comments thus: "The principal machinery used by Apollonius as well as by the earlier geometers comes under the head of what has been not inappropriately called a *geometrical algebra*."

The discoveries of Archimedes and Apollonius, says M. Chasles, marked the most brilliant epoch of ancient geometry. Two questions which have occupied geometers of all periods may be regarded as having originated with them. The first of these is the quadrature of curvilinear figures, which gave birth to the infinitesimal calculus. The second is the theory of conic sections, which was the prelude to the theory of geometrical curves of all degrees, and to that portion of geometry which considers only the forms and situations of figures and uses only the intersection of lines and surfaces and the ratios of rectilineal distances. These two great divisions of geometry may be designated by the names of *Geometry of Measurements* and *Geometry of Forms and Situations*, or, Geometry of Archimedes and of Apollonius.

Besides the *Conic Sections*, Pappus ascribes to Apollonius the following works: *On Contacts*, *Plane Loci*, *Inclinations*, *Section of an Area*, *Determinate Section*, and gives lemmas from which attempts have been made to restore the lost originals. Two books on *De Sectione Rationis* have been found in the Arabic. The book on *Contacts*, as restored by F. Vieta, contains the so-called "Apollonian Problem": Given three circles, to find a fourth which shall touch the three.

Euclid, Archimedes, and Apollonius brought geometry to as high a state of perfection as it perhaps could be brought without first introducing some more general and more powerful method than the old method of exhaustion. A briefer symbolism, a Cartesian geometry, an infinitesimal calculus, were needed. The Greek mind was not

[1] J. Gow, *op. cit.*, p. 252.

[2] T. L. Heath, *Apollonius of Perga*, edited in modern notation. Cambridge, 1896, p. ci.

adapted to the invention of general methods. Instead of a climb to still loftier heights we observe, therefore, on the part of later Greek geometers, a descent, during which they paused here and there to look around for details which had been passed by in the hasty ascent.[1]

Among the earliest successors of Apollonius was **Nicomedes.** Nothing definite is known of him, except that he invented the *conchoid* ("mussel-like"), a curve of the fourth order. He devised a little machine by which the curve could be easily described. With aid of the conchoid he duplicated the cube. The curve can also be used for trisecting angles in a manner resembling that in the eighth lemma of Archimedes. Proclus ascribes this mode of trisection to Nicomedes, but Pappus, on the other hand, claims it as his own. The conchoid was used by Newton in constructing curves of the third degree.

About the time of Nicomedes (say, 180 B. C.), flourished also **Diocles,** the inventor of the *cissoid* ("ivy-like"). This curve he used for finding two mean proportionals between two given straight lines. The Greeks did not consider the companion-curve to the cissoid; in fact, they considered only the part of the cissoid proper which lies inside the circle used in constructing the curve. The part of the area of the circle left over when the two circular areas on the concave sides of the branches of the curve are removed, looks somewhat like an ivy-leaf. Hence, probably, the name of the curve. That the two branches extend to infinity appears to have been noticed first by G. P. de Roberal in 1640 and then by R. de Sluse.[2]

About the life of **Perseus** we know as little as about that of Nicomedes and Diocles. He lived some time between 200 and 100 B. C. From Heron and Geminus we learn that he wrote a work on the *spire*, a sort of anchor-ring surface described by Heron as being produced by the revolution of a circle around one of its chords as an axis. The sections of this surface yield peculiar curves called *spiral sections*, which, according to Geminus, were thought out by Perseus. These curves appear to be the same as the *Hippopede* of Eudoxus.

Probably somewhat later than Perseus lived **Zenodorus.** He wrote an interesting treatise on a new subject; namely, *isoperimetrical figures.* Fourteen propositions are preserved by Pappus and Theon. Here are a few of them: Of isoperimetrical, regular polygons, the one having the largest number of angles has the greatest area; the circle has a greater area than any regular polygon of equal periphery; of all isoperimentrical polygons of n sides, the regular is the greatest; of all solids having surfaces equal in area, the sphere has the greatest volume.

Hypsicles (between 200 and 100 B. C.) was supposed to be the author of both the fourteenth and fifteenth books of Euclid, but recent critics are of opinion that the fifteenth book was written by an author

[1] M. Cantor, *op. cit.*, Vol. I, 3 Aufl., 1907, p. 350.
[2] G. Loria, *Ebene Curven*, transl. by F. Schütte, I, 1910, p. 37.

who lived several centuries after Christ. The fourteenth book contains seven elegant theorems on *regular solids*. A treatise of Hypsicles on *Risings* is of interest because it gives the division of the circumference into 360 degrees after the fashion of the Babylonians.

Hipparchus of Nicæa in Bithynia was the greatest astronomer of antiquity. He took astronomical observations between 161 and 127 B. C. He established inductively the famous theory of epicycles and eccentrics. As might be expected, he was interested in mathematics, not *per se*, but only as an aid to astronomical inquiry. No mathematical writings of his are extant, but Theon of Alexandria informs us that Hipparchus originated the science of *trigonometry*, and that he calculated a "table of chords" in twelve books. Such calculations must have required a ready knowledge of arithmetical and algebraical operations. He possessed arithmetical and also graphical devices for solving geometrical problems in a plane and on a sphere. He gives indication of having seized the idea of co-ordinate representation, found earlier in Apollonius.

About 100 B. C. flourished **Heron the Elder** of Alexandria. He was the pupil of Ctesibius, who was celebrated for his ingenious mechanical inventions, such as the hydraulic organ, the water-clock, and catapult. It is believed by some that Heron was a son of Ctesibius. He exhibited talent of the same order as did his master by the invention of the eolipile and a curious mechanism known as "Heron's fountain." Great uncertainty exists concerning his writings. Most authorities believe him to be the author of an important *Treatise on the Dioptra*, of which there exist three manuscript copies, quite dissimilar. But M. Marie[1] thinks that the *Dioptra* is the work of *Heron the Younger*, who lived in the seventh or eighth century after Christ, and that *Geodesy*, another book supposed to be by Heron, is only a corrupt and defective copy of the former work. *Dioptra* contains the important formula for finding the area of a triangle expressed in terms of its sides; its derivation is quite laborious and yet exceedingly ingenious. "It seems to me difficult to believe," says Chasles, "that so beautiful a theorem should be found in a work so ancient as that of Heron the Elder, without that some Greek geometer should have thought to cite it." Marie lays great stress on this silence of the ancient writers, and argues from it that the true author must be Heron the Younger or some writer much more recent than Heron the Elder. But no reliable evidence has been found that there actually existed a second mathematician by the name of Heron. P. Tannery has shown that, in applying this formula, Heron found the irrational square roots by the approximation, $\sqrt{A} \sim \frac{1}{2}(a + \frac{A}{a})$, where a^2 is the square nearest to

[1] Maximilien Marie, *Histoire des sciences mathématiques et physiques.* Paris, Tome I, 1883, p. 178.

A. When a more accurate value was wanted, Heron took $\frac{1}{2}(a+\frac{A}{a})$ in the place of a in the above formula. Apparently, Heron sometimes found square and cube roots also by the method of "double false position."

"Dioptra," says Venturi, were instruments which had great resemblance to our modern theodolites. The book *Dioptra* is a treatise on geodesy containing solutions, with aid of these instruments, of a large number of questions in geometry, such as to find the distance between two points, of which one only is accessible, or between two points which are visible but both inaccessible; from a given point to draw a perpendicular to a line which cannot be approached; to find the difference of level between two points; to measure the area of a field without entering it.

Heron was a practical surveyor. This may account for the fact that his writings bear so little resemblance to those of the Greek authors, who considered it degrading the science to apply geometry to surveying. The character of his geometry is not Grecian, but decidedly Egyptian. This fact is the more surprising when we consider that Heron demonstrated his familiarity with Euclid by writing a commentary on the *Elements*. Some of Heron's formulas point to an old Egyptian origin. Thus, besides the above exact formula for the area of a triangle in terms of its sides, Heron gives the formula $\frac{a_1+a_2}{2}\times\frac{b}{2}$, which bears a striking likeness to the formula $\frac{a_1+a_2}{2}\times\frac{b_1+b_2}{2}$ for finding the area of a quadrangle, found in the Edfu inscriptions. There are, moreover, points of resemblance between Heron's writings and the ancient Ahmes papyrus. Thus Ahmes used unit-fractions exclusively (except the fraction $\frac{2}{3}$); Heron uses them oftener than other fractions. Like Ahmes and the priests at Edfu, Heron divides complicated figures into simpler ones by drawing auxiliary lines; like them, he shows, throughout, a special fondness for the isosceles trapezoid.

The writings of Heron satisfied a practical want, and for that reason were borrowed extensively by other peoples. We find traces of them in Rome, in the Occident during the Middle Ages, and even in India.

The works attributed to Heron, including the newly discovered *Metrica* published in 1903, have been edited by J. H. Heiberg, H. Schöne and W. Schmidt.

Geminus of Rhodes (about 70 B. C.) published an astronomical work still extant. He wrote also a book, now lost, on the *Arrangement of Mathematics*, which contained many valuable notices of the early history of Greek mathematics. Proclus and Eutocius quote it frequently. Theodosius is the author of a book of little merit on the

geometry of the sphere. Investigations due to P. Tannery and A. A. Björnbo [1] seem to indicate that the mathematician Theodosius was not Theodosius of Tripolis, as formerly supposed, but was a resident of Bithynia and contemporary of Hipparchus. **Dionysodorus** of Amisus in Pontus applied the intersection of a parabola and hyperbola to the solution of a problem which Archimedes, in his *Sphere and Cylinder*, had left incomplete. The problem is "to cut a sphere so that its segments shall be in a given ratio."

We have now sketched the progress of geometry down to the time of Christ. Unfortunately, very little is known of the history of geometry between the time of Apollonius and the beginning of the Christian era. The names of quite a number of geometers have been mentioned, but very few of their works are now extant. It is certain, however, that there were no mathematicians of real genius from Apollonius to Ptolemy, excepting Hipparchus and perhaps Heron.

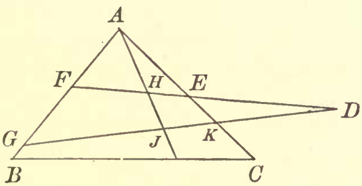
The Second Alexandrian School

The close of the dynasty of the Lagides which ruled Egypt from the time of Ptolemy Soter, the builder of Alexandria, for 300 years; the absorption of Egypt into the Roman Empire; the closer commercial relations between peoples of the East and of the West; the gradual decline of paganism and spread of Christianity,—these events were of far-reaching influence on the progress of the sciences, which then had their home in Alexandria. Alexandria became a commercial and intellectual emporium. Traders of all nations met in her busy streets, and in her magnificent Library, museums, lecture-halls, scholars from the East mingled with those of the West; Greeks began to study older literatures and to compare them with their own. In consequence of this interchange of ideas the Greek philosophy became fused with Oriental philosophy. Neo-Pythagoreanism and Neo-Platonism were the names of the modified systems. These stood, for a time, in opposition to Christianity. The study of Platonism and Pythagorean mysticism led to the revival of the theory of numbers. Perhaps the dispersion of the Jews and their introduction to Greek learning helped in bringing about this revival. The theory of numbers became a favorite study. This new line of mathematical inquiry ushered in what we may call a new school. There is no doubt that even now geometry continued to be one of the most important studies in the Alexandrian course. This Second Alexandrian School may be said to begin with the Christian era. It was made famous by the names of Claudius Ptolemæus, Diophantus, Pappus, Theon of Smyrna, Theon of Alexandria, Iamblichus, Porphyrius, and others.

By the side of these we may place **Serenus** of Antinœia, as having

[1] Axel Anthon Björnbo (1874–1911) of Copenhagen was a historian of mathematics. See *Bibliotheca mathematica*, 3 S., Vol. 12, 1911–12, pp. 337–344.

been connected more or less with this new school. He wrote on sections of the cone and cylinder, in two books, one of which treated only of the triangular section of the cone through the apex. He solved the problem, "given a cone (cylinder), to find a cylinder (cone), so that the section of both by the same plane gives similar ellipses." Of particular interest is the following theorem, which is the foundation of the modern theory of harmonics: If from D we draw DF, cutting the triangle ABC, and choose H on it, so that $DE : DF = EH : HF$, and if we draw the line AH, then every transversal through D, such as DG, will be divided by AH so that $DK : DG = KJ : JG$. **Menelaus** of Alexandria (about 98 A. D.) was the author of *Sphærica*, a work extant in Hebrew and Arabic, but not in Greek. In it he proves the



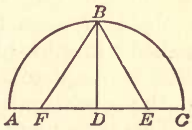
theorems on the congruence of spherical triangles, and describes their properties in much the same way as Euclid treats plane triangles. In it are also found the theorems that the sum of the three sides of a spherical triangle is less than a great circle, and that the sum of the three angles exceeds two right angles. Celebrated are two theorems of his on plane and spherical triangles. The one on plane triangles is that, "if the three sides be cut by a straight line, the product of the three segments which have no common extremity is equal to the product of the other three." L. N. M. Carnot makes this proposition, known as the "lemma of Menelaus," the base of his theory of transversals. The corresponding theorem for spherical triangles, the so-called "regula sex quantitatum," is obtained from the above by reading "chords of three segments doubled," in place of "three segments."

Claudius Ptolemy, a celebrated astronomer, was a native of Egypt. Nothing is known of his personal history except that he flourished in Alexandria in 139 A. D. and that he made the earliest astronomical observations recorded in his works, in 125 A. D., the latest in 151 A. D. The chief of his works are the *Syntaxis Mathematica* (or the *Almagest*, as the Arabs call it) and the *Geographica*, both of which are extant. The former work is based partly on his own researches, but mainly on those of Hipparchus. Ptolemy seems to have been not so much of an independent investigator, as a corrector and improver of the work of his great predecessors. The *Almagest* [1] forms the foundation of all astronomical science down to N. Copernicus. The fundamental idea of his system, the "Ptolemaic System," is that the earth is in the centre of the universe, and that the sun and planets revolve around the earth. Ptolemy did considerable for mathematics. He created,

[1] On the importance of the Almagest in the history of astronomy, consult P. Tannery, *Recherches sur l'histoire de l'astronomie*, Paris, 1893.

for astronomical use, a *trigonometry* remarkably perfect in form. The foundation of this science was laid by the illustrious Hipparchus.

The *Almagest* is in 13 books. Chapter 9 of the first book shows how to calculate tables of chords. The circle is divided into 360 degrees, each of which is halved. The diameter is divided into 120 divisions; each of these into 60 parts, which are again subdivided into 60 smaller parts. In Latin, these parts were called *partes minutæ primæ* and *partes minutæ secundæ*. Hence our names, "minutes" and "seconds." The sexagesimal method of dividing the circle is of Babylonian origin, and was known to Geminus and Hipparchus. But Ptolemy's method of calculating chords seems original with him. He first proved the proposition, now appended to Euclid VI (*D*), that "the rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides." He then shows how to find from the chords of two arcs the chords of their sum and difference, and from the chord of any arc that of its half. These theorems he applied to the calculation of his tables of chords. The proofs of these theorems are very pretty. Ptolemy's construction of sides of a regular inscribed pentagon and decagon was given later by C. Clavius and L. Mascheroni, and now is used much by engineers. Let the radius BD be $\perp$ to AC, $DE = EC$. Make $EF = EB$, then BF is the side of the pentagon and DF is the side of the decagon.



Another chapter of the first book in the *Almagest* is devoted to *trigonometry*, and to *spherical* trigonometry in particular. Ptolemy proved the "lemma of Menelaus," and also the "regula sex quantitatum." Upon these propositions he built up his trigonometry. In trigonometric computations, the Greeks did not use, as did the Hindus, half the chord of twice the arc (the "sine"); the Greeks used instead the whole chord of double the arc. Only in graphic constructions, referred to again later, did Ptolemy and his predecessors use half the chord of double the arc. The fundamental theorem of plane trigonometry, that two sides of a triangle are to each other as the chords of double the arcs measuring the angles opposite the two sides, was not stated explicitly by Ptolemy, but was contained implicitly in other theorems. More complete are the propositions in spherical trigonometry.

The fact that trigonometry was cultivated not for its own sake, but to aid astronomical inquiry, explains the rather startling fact that spherical trigonometry came to exist in a developed state earlier than plane trigonometry.

The remaining books of the *Almagest* are on astronomy. Ptolemy has written other works which have little or no bearing on mathematics, except one on geometry. Extracts from this book, made by Proclus, indicate that Ptolemy did not regard the parallel-axiom of

Euclid as self-evident, and that Ptolemy was the first of the long line of geometers from ancient time down to our own who toiled in the vain attempt to prove it. The untenable part of his demonstration is the assertion that, in case of parallelism, the sum of the interior angles on one side of a transversal must be the same as their sum on the other side of the transversal. Before Ptolemy an attempt to improve the theory of parallels was made by **Posidonius** (first cent. B. C.) who defined parallel lines as lines that are coplanar and equidistant. From an Arabic writer, *Al-Nirizi* (ninth cent.) it appears that Simplicius brought forward a proof of the 5th postulate, based upon this definition, and due to his friend Aganis (Geminus?).[1]

In the making of maps of the earth's surface and of the celestial sphere, Ptolemy (following Hipparchus) used stereographic projection. The eye is imagined to be at one of the poles, the projection being thrown upon the equatorial plane. He devised an instrument, a form of astrolabe planisphere, which is a stereographic projection of the celestial sphere.[2] Ptolemy wrote a monograph on the analemma which was a figure involving orthographic projections of the celestial sphere upon three mutually perpendicular planes (the horizontal, meridian and vertical circles). The analemma was used in determining positions of the sun, the rising and setting of the stars. The procedure was probably known to Hipparchus and the older astronomers. It furnished a graphic method for the solution of spherical triangles and was used subsequently by the Hindus, the Arabs, and Europeans as late as the seventeenth century.[3]

Two prominent mathematicians of this time were Nicomachus and Theon of Smyrna. Their favorite study was theory of numbers. The investigations in this science culminated later in the algebra of Diophantus. But no important geometer appeared after Ptolemy for 150 years. An occupant of this long gap was **Sextus Julius Africanus,** who wrote an unimportant work on geometry applied to the art of war, entitled *Cestes*. Another was the sceptic, **Sextus Empiricus** (200 A. D.); he endeavored to elucidate Zeno's "Arrow" by stating another argument equally paradoxical and therefore far from illuminating: Men never die, for if a man die, it must either be at a time when he is alive, or at a time when he is not alive; hence he never dies. Sextus Empiricus advanced also the paradox, that, when a line rotating in a plane about one of its ends describes a circle with each of its points, these concentric circles are of unequal area, yet each circle must be equal to the neighbouring circle which it touches.[1]

[1] R. Bonola, *Non-Euclidean Geometry*, trans. by H. S. Carslaw, Chicago, 1912, pp. 3–8. Robert Bonola (1875–1911) was professor in Rome.

[2] See M. Latham, "The Astrolabe," *Am. Math. Monthly*, Vol. 24, 1917, p. 162.

[3] See A. v. Braunmühl, *Geschichte der Trigonometrie*, Leipzig, I, 1900, p. 11. Alexander von Braunmühl (1853–1908) was professor at the technical high school in Munich.

Pappus, probably born about 340 A. D., in Alexandria, was the last great mathematician of the Alexandrian school. His genius was inferior to that of Archimedes, Apollonius, and Euclid, who flourished over 500 years earlier. But living, as he did, at a period when interest in geometry was declining, he towered above his contemporaries "like the peak of Teneriffa above the Atlantic." He is the author of a *Commentary on the Almagest*, a *Commentary on Euclid's Elements*, a *Commentary on the Analemma of Diodorus*,—a writer of whom nothing is known. All these works are lost. Proclus, probably quoting from the *Commentary on Euclid*, says that Pappus objected to the statement that an angle equal to a right angle is always itself a right angle.

The only work of Pappus still extant is his *Mathematical Collections*. This was originally in eight books, but the first and portions of the second are now missing. The *Mathematical Collections* seems to have been written by Pappus to supply the geometers of his time with a succinct analysis of the most difficult mathematical works and to facilitate the study of them by explanatory lemmas. But these lemmas are selected very freely, and frequently have little or no connection with the subject on hand. However, he gives very accurate summaries of the works of which he treats. The *Mathematical Collections* is invaluable to us on account of the rich information it gives on various treatises by the foremost Greek mathematicians, which are now lost. Mathematicians of the last century considered it possible to restore lost work from the *résumé* by Pappus alone.

We shall now cite the more important of those theorems in the *Mathematical Collections* which are supposed to be original with Pappus. First of all ranks the elegant theorem re-discovered by *P. Guldin*, over 1000 years later, that the volume generated by the revolution of a plane curve which lies wholly on one side of the axis, equals the area of the curve multiplied by the circumference described by its center of gravity. Pappus proved also that the centre of gravity of a triangle is that of another triangle whose vertices lie upon the sides of the first and divide its three sides in the same ratio. In the fourth book are new and brilliant propositions on the quadratrix which indicate an intimate acquaintance with curved surfaces. He generates the quadratrix as follows: Let a spiral line be drawn upon a right circular cylinder; then the perpendiculars to the axis of the cylinder drawn from each point of the spiral line form the surface of a screw. A plane passed through one of these perpendiculars, making any convenient angle with the base of the cylinder, cuts the screw-surface in a curve, the orthogonal projection of which upon the base is the *quadratrix*. A second mode of generation is no less admirable: If we make the spiral of Archimedes the base of a right

[1] See K. Lasswitz, *Geschichte der Atomistik*, I, Hamburg und Leipzig, 1890, p. 148.

cylinder, and imagine a cone of revolution having for its axis the side of the cylinder passing through the initial point of the spiral, then this cone cuts the cylinder in a curve of double curvature. The perpendiculars to the axis drawn through every point in this curve form the surface of a screw which Pappus here calls the *plectoidal surface.* A plane passed through one of the perpendiculars at any convenient angle cuts that surface in a curve whose orthogonal projection upon the plane of the spiral is the required *quadratrix.* Pappus considers curves of double curvature still further. He produces a *spherical spiral* by a point moving uniformly along the circumference of a great circle of a sphere, while the great circle itself revolves uniformly around its diameter. He then finds the area of that portion of the surface of the sphere determined by the spherical spiral, "a complanation which claims the more lively admiration, if we consider that, although the entire surface of the sphere was known since Archimedes' time, to measure portions thereof, such as spherical triangles, was then and for a long time afterwards an unsolved problem." [1] A question which was brought into prominence by Descartes and Newton is the "problem of Pappus." Given several straight lines in a plane, to find the locus of a point such that when perpendiculars (or, more generally, straight lines at given angles) are drawn from it to the given lines, the product of certain ones of them shall be in a given ratio to the product of the remaining ones. It is worth noticing that it was Pappus who first found the focus of the parabola and propounded the theory of the involution of points. He used the directrix and was the first to put in definite form the definition of the conic sections as loci of those points whose distances from a fixed point and from a fixed line are in a constant ratio. He solved the problem to draw through three points lying in the same straight line, three straight lines which shall form a triangle inscribed in a given circle. From the *Mathematical Collections* many more equally difficult theorems might be quoted which are original with Pappus as far as we know. It ought to be remarked, however, that he has been charged in three instances with copying theorems without giving due credit, and that he may have done the same thing in other cases in which we have no data by which to ascertain the real discoverer.[2]

About the time of Pappus lived **Theon** of Alexandria. He brought out an edition of Euclid's *Elements* with notes, which he probably used as a text-book in his classes. His commentary on the *Almagest* is valuable for the many historical notices, and especially for the specimens of Greek arithmetic which it contains. Theon's daughter **Hypatia,** a woman celebrated for her beauty and modesty, was the last Alexandrian teacher of reputation, and is said to have been an

[1] M. Cantor, *op. cit.*, Vol. I, 3 Aufl., 1907, p. 451.

[2] For a defence of Pappus against these charges, see J. H. Weaver in *Bull. Am Math. Soc.*, Vol. 23, 1916, pp. 131–133.

abler philosopher and mathematician than her father. Her notes on the works of Diophantus and Apollonius have been lost. Her tragic death in 415 A. D. is vividly described in Kingsley's *Hypatia.*

From now on, mathematics ceased to be cultivated in Alexandria. The leading subject of men's thoughts was Christian theology. Paganism disappeared, and with it pagan learning. The Neo-Platonic school at Athens struggled on a century longer. Proclus, Isidorus, and others kept up the "golden chain of Platonic succession." **Proclus,** the successor of Syrianus, at the Athenian school, wrote a commentary on Euclid's *Elements.* We possess only that on the first book, which is valuable for the information it contains on the history of geometry. **Damascius** of Damascus, the pupil of Isidorus, is now believed to be the author of the fifteenth book of Euclid. Another pupil of Isidorus was **Eutocius** of Ascalon, the commentator of Apollonius and Archimedes. **Simplicius** wrote a commentary on Aristotle's *De Cœlo.* Simplicius reports Zeno as saying: "That which, being added to another, does not make it greater, and being taken away from another does not make it less, is nothing." According to this, the denial of the existence of the infinitesimal goes back to Zeno. This momentous question presented itself centuries later to Leibniz, who gave different answers. The report made by Simplicius of the quadratures of Antiphon and Hippocrates of Chios is one of the best sources of historical information on this point.[1] In the year 529, Justinian, disapproving heathen learning, finally closed by imperial edict the schools at Athens.

As a rule, the geometers of the last 500 years showed a lack of creative power. They were commentators rather than discoverers.

The principal characteristics of ancient geometry are:—

(1) A wonderful clearness and definiteness of its concepts and an almost perfect logical rigor of its conclusions.

(2) A complete want of general principles and methods. Ancient geometry is decidedly *special.* Thus the Greeks possessed no general method of drawing tangents. "The determination of the tangents to the three conic sections did not furnish any rational assistance for drawing the tangent to any other new curve, such as the conchoid, the cissoid, etc." In the demonstration of a theorem, there were, for the ancient geometers, as many different cases requiring separate proof as there were different positions for the lines. The greatest geometers considered it necessary to treat all possible cases independently of each other, and to prove each with equal fulness. To devise methods by which the various cases could all be disposed of by one stroke, was beyond the power of the ancients. "If we compare a mathematical problem with a huge rock, into the interior of which we desire to penetrate, then the work of the Greek mathe-

[1] See F. Rudio in *Bibliotheca mathematica,* 3 S., Vol. 3, 1902, pp. 7–62.

maticians appears to us like that of a vigorous stonecutter who, with chisel and hammer, begins with indefatigable perseverance, from without, to crumble the rock slowly into fragments; the modern mathematician appears like an excellent miner, who first bores through the rock some few passages, from which he then bursts it into pieces with one powerful blast, and brings to light the treasures within." [1]

Greek Arithmetic and Algebra

Greek mathematicians were in the habit of discriminating between the *science* of numbers and the *art* of calculation. The former they called *arithmetica*, the latter *logistica*. The drawing of this distinction between the two was very natural and proper. The difference between them is as marked as that between theory and practice. Among the Sophists the art of calculation was a favorite study. Plato, on the other hand, gave considerable attention to philosophical arithmetic, but pronounced calculation a vulgar and childish art.

In sketching the history of Greek calculation, we shall first give a brief account of the Greek mode of counting and of writing numbers. Like the Egyptians and Eastern nations, the earliest Greeks counted on their fingers or with pebbles. In case of large numbers, the pebbles were probably arranged in parallel vertical lines. Pebbles on the first line represented units, those on the second tens, those on the third hundreds, and so on. Later, frames came into use, in which strings or wires took the place of lines. According to tradition, Pythagoras, who travelled in Egypt and, perhaps, in India, first introduced this valuable instrument into Greece. The *abacus*, as it is called, existed among different peoples and at different times, in various stages of perfection. An abacus is still employed by the Chinese under the name of *Swan-pan*. We possess no specific information as to how the Greek abacus looked or how it was used. Boethius says that the Pythagoreans used with the abacus certain nine signs called *apices*, which resembled in form the nine "Arabic numerals." But the correctness of this assertion is subject to grave doubts.

The oldest Grecian numerical symbols were the so-called *Herodianic signs* (after Herodianus, a Byzantine grammarian of about 200 A. D., who describes them). These signs occur frequently in Athenian inscriptions and are, on that account, now generally called *Attic*. For some unknown reason these symbols were afterwards replaced by the *alphabetic numerals*, in which the letters of the Greek alphabet were used, together with three strange and antique letters ϛ, ϙ, and ϡ, and the symbol M. This change was decidedly for the worse, for the old Attic numerals were less burdensome on the memory, inasmuch

[1] H. Hankel, *Die Entwickelung der Mathematik in den letzten Jahrhunderten.* Tübingen, 1884, p. 16.

as they contained fewer symbols and were better adapted to show forth analogies in numerical operations. The following table shows the Greek alphabetic numerals and their respective values:—

α	β	γ	δ	ϵ	ϛ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	o	π	ϙ
1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90

ρ	σ	τ	υ	ϕ	χ	ψ	ω	ϡ	$,\alpha$	$,\beta$	$,\gamma$ etc.
100	200	300	400	500	600	700	800	900	1000	2000	3000

M	$\overset{\beta}{M}$	$\overset{\gamma}{M}$ etc.
10,000	20,000	30,000

It will be noticed that at 1000, the alphabet is begun over again, but, to prevent confusion, a stroke is now placed before the letter and generally somewhat below it. A horizontal line drawn over a number served to distinguish it more readily from words. The coefficient for M was sometimes placed before or behind instead of over the M. Thus 43,678 was written $\delta\overline{M,\gamma\chi o\eta}$. It is to be observed that the Greeks had no zero.

Fractions were denoted by first writing the numerator marked with an accent, then the denominator marked with two accents and written twice. Thus, $\iota\gamma'\kappa\theta''\kappa\theta''=\frac{13}{29}$. In case of fractions having unity for the numerator, the α' was omitted and the denominator was written only once. Thus $\mu\delta''=\frac{1}{44}$.

The Greeks had the name epimorion for the ratio $\frac{n}{n+1}$. Archytas proved the theorem that if an epimorion $\frac{\alpha}{\beta}$ is reduced to its lowest terms $\frac{\mu}{\nu}$, then $\nu=\mu+1$. This theorem is found later in the musical writings of Euclid and of the Roman Boethius. The Euclidean form of arithmetic, without perhaps the representation of numbers by lines, existed as early as the time of Archytas.[1]

Greek writers seldom refer to calculation with alphabetic numerals. Addition, subtraction, and even multiplication were probably performed on the abacus. Expert mathematicians may have used the symbols. Thus Eutocius, a commentator of the sixth century after Christ, gives a great many multiplications of which the following is a specimen:[2]—

[1] P. Tannery in *Bibliotheca mathematica*, 3 S., Vol. VI, 1905, p. 228.

[2] J. Gow, *op. cit.*, p. 50.

$\overline{\sigma\xi\epsilon}$	2 6 5
$\overline{\sigma\xi\epsilon}$	2 6 5
$\overset{\delta}{\mathrm{M}}\overset{\alpha}{\mathrm{M}},\beta,\alpha$	40000, 12000, 1000
$\overset{\alpha}{\mathrm{M}},\beta,\overline{\gamma\chi\tau}$	12000, 3600, 300
$,\alpha\;\overline{\tau\;\kappa\epsilon}$	1000, 300, 25
$\overset{\zeta}{\mathrm{M}}\;\overline{\sigma\kappa\epsilon}$	70225

The operation is explained sufficiently by the modern numerals appended. In case of mixed numbers, the process was still more clumsy. Divisions are found in Theon of Alexandria's commentary on the *Almagest*. As might be expected, the process is long and tedious.

We have seen in geometry that the more advanced mathematicians frequently had occasion to extract the square root. Thus Archimedes in his *Mensuration of the Circle* gives a large number of square roots. He states, for instance, that $\sqrt{3}<\frac{1351}{780}$ and $\sqrt{3}>\frac{265}{153}$, but he gives no clue to the method by which he obtained these approximations. It is not improbable that the earlier Greek mathematicians found the square root by trial only. Eutocius says that the method of extracting it was given by Heron, Pappus, Theon, and other commentators on the *Almagest*. Theon's is the only one of these methods known to us. It is the same as the one used nowadays, except that sexagesimal fractions are employed in place of our decimals. What the mode of procedure actually was when sexagesimal fractions were not used, has been the subject of conjecture on the part of numerous modern writers.

Of interest, in connection with arithmetical symbolism, is the *Sand-Counter* (Arenarius), an essay addressed by **Archimedes** to Gelon, king of Syracuse. In it Archimedes shows that people are in error who think the sand cannot be counted, or that if it can be counted, the number cannot be expressed by arithmetical symbols. He shows that the number of grains in a heap of sand not only as large as the whole earth, but as large as the entire universe, can be arithmetically expressed. Assuming that 10,000 grains of sand suffice to make a little solid of the magnitude of a poppy-seed, and that the diameter of a poppy-seed be not smaller than $\frac{1}{40}$ part of a finger's breadth; assuming further, that the diameter of the universe (supposed to extend to the sun) be less than 10,000 diameters of the earth, and that the latter be less than 1,000,000 stadia, Archimedes finds a number which would exceed the number of grains of sand in the sphere of the universe. He goes on even further. Supposing the universe to reach out to the fixed stars, he finds that the sphere, having the distance from the earth's centre to the fixed stars for its radius, would contain a number of grains of sand less than 1000 myriads of the eighth octad. In our notation, this number would be 10^{63} or 1 with 63 ciphers after it. It can hardly be doubted that one object which Archimedes had in view in making this calculation was the improvement of the Greek symbolism. It is not known whether he invented some short notation by which to represent the above number or not.

We judge from fragments in the second book of Pappus that Apollonius proposed an improvement in the Greek method of writing numbers, but its nature we do not know. Thus we see that the Greeks never possessed the boon of a clear, comprehensive symbolism. The honor of giving such to the world was reserved by the irony of fate for a nameless Indian of an unknown time, and we know not whom to thank for an invention of such importance to the general progress of intelligence.[1]

Passing from the subject of *logistica* to that of *arithmetica*, our attention is first drawn to the science of numbers of **Pythagoras.** Before founding his school, Pythagoras studied for many years under the Egyptian priests and familiarised himself with Egyptian mathematics and mysticism. If he ever was in Babylon, as some authorities claim, he may have learned the sexagesimal notation in use there; he may have picked up considerable knowledge on the theory of proportion, and may have found a large number of interesting astronomical observations. Saturated with that speculative spirit then pervading the Greek mind, he endeavored to discover some principle of homogeneity in the universe. Before him, the philosophers of the Ionic school had sought it in the matter of things; Pythagoras looked for it in the structure of things. He observed various numerical relations or analogies between numbers and the phenomena of the universe. Being convinced that it was in numbers and their relations that he was to find the foundation to true philosophy, he proceeded to trace the origin of all things to numbers. Thus he observed that musical strings of equal length stretched by weights having the proportion of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, produced intervals which were an octave, a fifth, and a fourth. Harmony, therefore, depends on musical proportion; it is nothing but a mysterious numerical relation. Where harmony is, there are numbers. Hence the order and beauty of the universe have their origin in numbers. There are seven intervals in the musical scale, and also seven planets crossing the heavens. The same numerical relations which underlie the former must underlie the latter. But where numbers are, there is harmony. Hence his spiritual ear discerned in the planetary motions a wonderful "harmony of the spheres." The Pythagoreans invested particular numbers with extraordinary attributes. Thus *one* is the essence of things; it is an absolute number; hence the origin of all numbers and so of all things. *Four* is the most perfect number, and was in some mystic way conceived to correspond to the human soul. Philolaus believed that 5 is the cause of color, 6 of cold, 7 of mind and health and light, 8 of love and friendship.[2] In Plato's works are evidences of a similar belief in religious relations of numbers. Even Aristotle referred the virtues to numbers.

Enough has been said about these mystic speculations to show what lively interest in mathematics they must have created and

[1] J. Gow, *op. cit.*, p. 63.

[2] J. Gow, *op. cit.*, p. 69.

maintained. Avenues of mathematical inquiry were opened up by them which otherwise would probably have remained closed at that time.

The Pythagoreans classified numbers into odd and even. They observed that the sum of the series of odd numbers from 1 to $2n+1$ was always a complete square, and that by addition of the even numbers arises the series 2, 6, 12, 20, in which every number can be decomposed into two factors differing from each other by unity. Thus, $6=2.3$, $12=3.4$, etc. These latter numbers were considered of sufficient importance to receive the separate name of *heteromecic* (not equilateral). Numbers of the form $\frac{n(n+1)}{2}$ were called *triangular*, because they could always be arranged thus,  Numbers which were equal to the sum of all their possible factors, such as 6, 28, 496, were called *perfect;* those exceeding that sum, *excessive;* and those which were less, *defective. Amicable* numbers were those of which each was the sum of the factors in the other. Much attention was paid by the Pythagoreans to the subject of proportion. The quantities a, b, c, d were said to be in *arithmetical* proportion when $a-b=c-d$; in *geometrical* proportion, when $a:b=c:d$; in *harmonic* proportion, when $a-b:b-c=a:c$. It is probable that the Pythagoreans were also familiar with the *musical* proportion $a:\frac{a+b}{2}=\frac{2ab}{a+b}:b$. Iamblichus says that Pythagoras introduced it from Babylon.

In connection with arithmetic, Pythagoras made extensive investigations into geometry. He believed that an arithmetical fact had its analogue in geometry, and *vice versa*. In connection with his theorem on the right triangle he devised a rule by which integral numbers could be found, such that the sum of the squares of two of them equalled the square of the third. Thus, take for one side an odd number $(2n+1)$; then $\frac{(2n+1)^2-1}{2}=2n^2+2n=$ the other side, and $(2n^2+2n+1)=$ hypotenuse. If $2n+1=9$, then the other two numbers are 40 and 41. But this rule only applies to cases in which the hypotenuse differs from one of the sides by 1. In the study of the right triangle there doubtless arose questions of puzzling subtlety. Thus, given a number equal to the side of an isosceles right triangle, to find the number which the hypotenuse is equal to. The side may have been taken equal to 1, 2, $\frac{3}{2}$, $\frac{6}{5}$, or any other number, yet in every instance all efforts to find a number exactly equal to the hypotenuse must have remained fruitless. The problem may have been attacked again and again, until finally "some rare genius, to whom it is granted, during some happy moments, to soar with eagle's flight above the level of human thinking," grasped the happy thought that this prob-

lem cannot be solved. In some such manner probably arose the theory of *irrational quantities*, which is attributed by Eudemus to the Pythagoreans. It was indeed a thought of extraordinary boldness, to assume that straight lines could exist, differing from one another not only in length,—that is, in quantity,—but also in a quality, which, though real, was absolutely invisible.[1] Need we wonder that the Pythagoreans saw in irrationals a deep mystery, a symbol of the unspeakable? We are told that the one who first divulged the theory of irrationals, which the Pythagoreans kept secret, perished in consequence in a shipwreck, "for the unspeakable and invisible should always be kept secret." Its discovery is ascribed to Pythagoras, but we must remember that all important Pythagorean discoveries were, according to Pythagorean custom, referred back to him. The first incommensurable ratio known seems to have been that of the side of a square to its diagonal, as $1 : \sqrt{2}$. **Theodorus of Cyrene** added to this the fact that the sides of squares represented in length by $\sqrt{3}$, $\sqrt{5}$, etc., up to $\sqrt{17}$, and Theætetus, that the sides of any square, represented by a surd, are incommensurable with the linear unit. **Euclid** (about 300 B. C.), in his *Elements*, X, 9, generalised still further: Two magnitudes whose squares are (or are not) to one another as a square number to a square number are commensurable (or incommensurable), and conversely. In the tenth book, he treats of incommensurable quantities at length. He investigates every possible variety of lines which can be represented by $\sqrt{\sqrt{a} \pm \sqrt{b}}$, a and b representing two commensurable lines, and obtains 25 species. Every individual of every species is incommensurable with all the individuals of every other species. "This book," says De Morgan, "has a completeness which none of the others (not even the fifth) can boast of; and we could almost suspect that Euclid, having arranged his materials in his own mind, and having completely elaborated the tenth book, wrote the preceding books after it, and did not live to revise them thoroughly." [2] The theory of incommensurables remained where Euclid left it, till the fifteenth century.

If it be recalled that the early Egyptians had some familiarity with quadratic equations, it is not surprising if similar knowledge is displayed by Greek writers in the time of Pythagoras. Hippocrates, in the fifth century B. C., when working on the areas of lunes, assumes the geometrical equivalent of the solution of the quadratic equation $x^2 + \sqrt{\frac{3}{2}}\, ax = a^2$. The complete geometrical solution was given by Euclid in his *Elements*, VI, 27–29. He solves certain types of quadratic equations geometrically in Book II, 5, 6, 11.

[1] H. Hankel, *Zur Geschichte der Mathematik in Mittelalter und Alterthum*, 1874, p. 102.

[2] A. De Morgan, "Eucleides" in *Smith's Dictionary of Greek and Roman Biog. and Myth.*

Euclid devotes the seventh, eighth, and ninth books of his *Elements* to arithmetic. Exactly how much contained in these books is Euclid's own invention, and how much is borrowed from his predecessors, we have no means of knowing. Without doubt, much is original with Euclid. The *seventh book* begins with twenty-one definitions. All except that for "prime" numbers are known to have been given by the Pythagoreans. Next follows a process for finding the G. C. D. of two or more numbers. The *eighth book* deals with numbers in continued proportion, and with the mutual relations of squares, cubes, and plane numbers. Thus, XXII, if three numbers are in continued proportion, and the first is a square, so is the third. In the *ninth book*, the same subject is continued. It contains the proposition that the number of primes is greater than any given number.

After the death of Euclid, the theory of numbers remained almost stationary for 400 years. Geometry monopolised the attention of all Greek mathematicians. Only two are known to have done work in arithmetic worthy of mention. **Eratosthenes** (275–194 B. C.) invented a "sieve" for finding prime numbers. All composite numbers are "sifted" out in the following manner: Write down the odd numbers from 3 up, in succession. By striking out every third number after the 3, we remove all multiples of 3. By striking out every fifth number after the 5, we remove all multiples of 5. In this way, by rejecting multiples of 7, 11, 13, etc., we have left prime numbers only. **Hypsicles** (between 200 and 100 B. C.) worked at the subjects of polygonal numbers and arithmetical progressions, which Euclid entirely neglected. In his work on "risings of the stars," he showed (1) that in an arithmetical series of $2n$ terms, the sum of the last n terms exceeds the sum of the first n by a multiple of n^2; (2) that in such a series of $2n+1$ terms, the sum of the series is the number of terms multiplied by the middle term; (3) that in such a series of $2n$ terms, the sum is half the number of terms multiplied by the two middle terms.[1]

For two centuries after the time of Hypsicles, arithmetic disappears from history. It is brought to light again about 100 A. D. by **Nicomachus,** a Neo-Pythagorean, who inaugurated the final era of Greek mathematics. From now on, arithmetic was a favorite study, while geometry was neglected. Nicomachus wrote a work entitled *Introductio Arithmetica*, which was very famous in its day. The great number of commentators it has received vouch for its popularity. Boethius translated it into Latin. Lucian could pay no higher compliment to a calculator than this: "You reckon like Nicomachus of Gerasa." The *Introductio Arithmetica* was the first exhaustive work in which arithmetic was treated quite independently of geometry. Instead of drawing lines, like Euclid, he illustrates things by real numbers. To be sure, in his book the old geometrical nomenclature is retained, but the method is inductive instead of deductive. "Its sole

[1] J. Gow, *op. cit.*, p. 87.

business is classification, and all its classes are derived from, and exhibited by, actual numbers." The work contains few results that are really original. We mention one important proposition which is probably the author's own. He states that cubical numbers are always equal to the sum of successive odd numbers. Thus, $8=2^3=3+5$, $27=3^3=7+9+11$, $64=4^3=13+15+17+19$, and so on. This theorem was used later for finding the sum of the cubical numbers themselves. **Theon** of Smyrna is the author of a treatise on "the mathematical rules necessary for the study of Plato." The work is ill arranged and of little merit. Of interest is the theorem, that every square number, or that number minus 1, is divisible by 3 or 4 or both. A remarkable discovery is a proposition given by **Iamblichus** in his treatise on Pythagorean philosophy. It is founded on the observation that the Pythagoreans called 1, 10, 100, 1000, units of the first, second, third, fourth "course" respectively. The theorem is this: If we add any three consecutive numbers, of which the highest is divisible by 3, then add the digits of that sum, then, again, the digits of *that* sum, and so on, the final sum will be 6. Thus, $61+62+63=186$, $1+8+6=15$, $1+5=6$. This discovery was the more remarkable, because the ordinary Greek numerical symbolism was much less likely to suggest any such property of numbers than our "Arabic" notation would have been.

Hippolytus, who appears to have been bishop at Portus Romae in Italy in the early part of the third century, must be mentioned for the giving of "proofs" by casting out the 9's and the 7's.

The works of Nicomachus, Theon of Smyrna, Thymaridas, and others contain at times investigations of subjects which are really algebraic in their nature. Thymaridas in one place uses the Greek, word meaning "unknown quantity" in a way which would lead one to believe that algebra was not far distant. Of interest in tracing the invention of algebra are the arithmetical epigrams in the *Palatine Anthology*, which contain about fifty problems leading to linear equations. Before the introduction of algebra these problems were propounded as puzzles. A riddle attributed to Euclid and contained in the *Anthology* is to this effect: A mule and a donkey were walking along, laden with corn. The mule says to the donkey, "If you gave me one measure, I should carry twice as much as you. If I gave you one, we should both carry equal burdens. Tell me their burdens, O most learned master of geometry." [1]

It will be allowed, says Gow, that this problem, if authentic, was not beyond Euclid, and the appeal to geometry smacks of antiquity. A far more difficult puzzle was the famous "cattle-problem," which Archimedes propounded to the Alexandrian mathematicians. The problem is indeterminate, for from only seven equations, eight unknown quantities in integral numbers are to be found. It may be

[1] J. Gow, *op. cit.*, p. 99.